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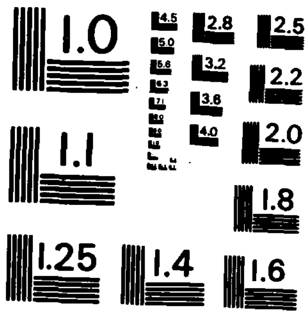
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CONTENTS

Special Issue on the Singularity Expansion Method
L. WILSON PEARSON and LENNART MARIN, Guest Editors

Contents (*Continued*)

INTRODUCTION

This special issue of Electromagnetics is dedicated to the subject of the Singularity Expansion Method (SEM) - in particular the mathematical aspects of SEM. In fact, the issue forms the proceedings of a meeting "Mathematical Foundations of the Singularity Expansion Method" held at the Carnahan House of the University of Kentucky in November 1980 under the sponsorship of the Air Force Office of Scientific Research (AFOSR). The purpose of the meeting was to bring together a group of mathematicians and engineers who have worked on different aspects of the SEM to foster interdisciplinary communication between the groups. The hope, of course, was that this communication might lead to the resolution of some questions regarding the mathematical rigor that have persisted throughout the development of the SEM. This communication we believe certainly led to a better understanding between the two groups of what the important questions are and the available means of attacking these questions.

Ten years have passed since Carl Baum first formalized the SEM as a means of treating transient and broad band electromagnetic scattering problems[3.1].* This development was sparked by the results from many experiments where different scatterers were exposed to transient electromagnetic fields. It was observed during these experiments that the response of the scatterer appeared to consist of a superposition of damped sinusoidal oscillations whose frequencies are related to the size of the scatterer. The natural question that arose was: "Is it possible to express any external scattering response as a sum of damped oscillations whose resonances and damping constants only depend on the scatterer itself, much in the same way as one can construct the response of a cavity?" The SEM was developed when trying to answer this question.

Much work during the last ten years has gone into trying to put the SEM on a solid mathematical foundation and applying it to various scattering problems. Workers who have tried to solidify the mathematical foundations for the method have found a great deal of frustration in dealing with such issues as space-time problems, nonself-adjoint operators, and analytic function theory. There are few general mathematical results which define the SEM representation within the confines of well defined mathematical and physical constraints. In many cases, workers have had to make whatever observations they can from the solution of a specific problem and then extend these results

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*A bibliography on SEM is included at the end of this issue and is shared in common by the papers herein.

using their physical/mathematical intuition. The wealth of semiempirical data acquired this way nevertheless have resulted in heuristically derived rules for the applicability and validity of the SEM. Thus, even in the face of the persistent difficulties in developing general theory, SEM stands as a powerful tool in electromagnetic and acoustic scattering theory.

The strength of the SEM primarily rests with the fact that both transient and time harmonic scattering quantities can be represented as a sum of conveniently factored products. One factor in this product depends only on the scatterer itself whereas the other depends on the exciting (or incident field.) The quantities that enter into the object-dependent factor are the object's complex resonant frequencies and the associated natural mode currents. The constellation of natural frequencies can be used to characterize the scattering object, thus opening the possibility of using SEM for target classification purposes. The expansion of the object's response in terms of natural modes allows for a circuit description of certain EM properties of the object.

The discussions during the meeting in the Carnahan House reflected the differences in the mathematicians's and engineer's outlooks. A mathematician participant was careful to categorize his comments into "results" (conclusions which can be mathematically proven) and "observations" (conclusions drawn from special cases but not proven mathematically). Engineers were quick to state that a significant part of their SEM related activity is predicated upon "observations" only (as is so much of their overall work). As a consequence the papers contained in this issue can perhaps be described as a collection of "results" and "observations." We leave it to the reader to distinguish between "results" and "observations" and the relative merit of the two.

We wish to thank C. L. Dolph for his help in planning the SEM meeting and R. N. Buchal of AFOSR for his support of and interest in the meeting. The assistance and support of the College of Engineering of the University of Kentucky and its Office of Continuing Education are also gratefully acknowledged. The bibliography at the end of this issue was prepared by Krzysztof A. Michalski.

L. Wilson Pearson and Lennart Marin
Guest Editors

THE SINGULARITY EXPANSION METHOD: BACKGROUND AND DEVELOPMENTS

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ABSTRACT

The singularity expansion method (SEM) arose from the observation that the transient response of complex electromagnetic scatterers appeared to be dominated by a small number of damped sinusoids. In the complex frequency plane, these damped sinusoids are poles of the Laplace-transformed response. The question is then one of characterizing the object response (time and frequency domains) in terms of all the singularities (poles, branch cuts, entire functions) in the complex frequency plane (hence singularity expansion method). Building on the older concept of natural frequencies, formulae were developed for the pole terms from an integral-equation formulation of the scattering process. The resulting factoring of the pole terms has important application consequences. Later developments include the eigenmode expansion method (EEM) which diagonalizes the integral-equation kernels and which can be used as an intermediate step in ordering the SEM terms. Additional concepts which have appeared include eigenimpedance synthesis and equivalent electrical networks. Of current interest is the use of the theoretical formulae to efficiently analyze and order experimental data. Related to this is the application of SEM results to target identification. This paper does not delve into the mathematical details; it presents an overview of the history and major concepts and results in SEM and EEM and related matters.

1. BACKGROUND

1.1 Natural Frequencies

An important antecedent physical concept is that of natural frequencies. These are thought of as frequencies for which there is a response with no forcing function. Also called natural oscillations or resonances, these in general exhibit a damping phenomenon (in the case of passive objects) which can be interpreted as one part of a complex frequency. In electromagnetic responses of various scatterers/antennas, there are various examples of early work on natural frequencies. The perfectly conducting thin wire and circular loop were treated by numerous investigators including Pocklington in 1897 [1.14]*, Abraham [1.1,1.2], Oseen [1.6-1.9], Hallén [1.5], and Rayleigh [1.15,1.16]. This was extended to perfectly conducting prolate spheroids by Page and Adams [1.10-1.13] and perfectly conducting spheres by Stratton [1.21]. An important

*Citation numbers refer to the collected bibliography appearing elsewhere in this issue.

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0272-6343/81/040351-10$2.25

contribution was made by Schwinger [6.89] who treated the special case of electromagnetic fields internal to perfectly conducting cavities. In this case the natural frequencies are all on the $j\omega$ axis (pure imaginary) in the complex-frequency (s) or Laplace-transform plane, and the natural modes have a convenient orthogonality property.

1.2 Laplace Transform

Various mathematical tools had been in use in electrical engineering and provided some starting point for constructing basic SEM formulae when the time was ripe. One such tool was certainly the Laplace (or Fourier) transform which we take in the two-sided sense as

$$\tilde{F}(s) \equiv \int_{-\infty}^{\infty} F(t)\, e^{-st}\, dt \quad , \quad F(t) = \frac{1}{2\pi j}\int_{\Omega_0 - j\infty}^{\Omega_0 + j\infty} \tilde{F}(s)\, e^{st}\, ds$$

$$t \equiv \text{time} \quad , \quad \tilde{\ } \text{ (above)} \equiv \text{Laplace transform}$$

$$s \equiv \Omega + j\omega = \text{Laplace transform variable} = \text{complex frequency} \tag{1.1}$$

$F(t) \equiv$ any Laplace transformable time function or operator (scalar, vector, tensor, etc.)

where the Bromwich contour, $\text{Re}[s] = \Omega_0$, for inversion is chosen in the strip of convergence, say $\Omega_a < \text{Re}[s] < \Omega_b$.

1.3 Complex Variable Theory

Considering the response of some antenna or scatterer as a function of s in the complex s plane one can describe the s-plane behavior in terms of the singularities (or boundaries of analyticity) in the complex plane, including the behavior at infinity (entire function). Appropriate contour integrals can be used to describe the response; the contours can be deformed to give separate terms for each singularity in both complex-frequency and time domains [2.3].

1.4 Circuit and System Theory

In electrical engineering there has been a considerable body of knowledge developed concerning electrical networks. This is summarized in circuit analysis and circuit synthesis theory which (especially in the linear case) is documented in numerous texts. This is further extended to linear system theory and control theory which are now major subject areas with an extensive literature. The use of the Laplace transform is quite extensive in these areas, and expansions in terms of poles are often used. Our problem of electromagnetic interaction (scattering) is related in that a scatterer can be thought of as a distributed network or system of a special kind (with response described by the Maxwell equations). Furthermore, it is possible to describe the scattering process by an equivalent circuit by using circuit synthesis concepts to synthesize (perhaps approximately) the appropriate complex transfer functions and impedances of the scatterer.

1.5 Integral Equations for Electromagnetic Scatterers and Antennas

For perfectly conducting objects (as well as for certain types of impedance loading) an integral equation reduces the problem from three space dimensions (the Maxwell differential equations) to two dimensions (the scatterer surface). Perhaps more important, the radiation condition at infinity for the scattered fields is explicitly incorporated into the integral equation so that one need not be concerned with the analytic continuation of the radiation condition into the left half of the s plane. Well-known integral equations include the electric-field integral equation and the magnetic-field integral equation (in various forms). In one-dimensional approximate forms (for wires) there are the Hallén and Pocklington equations. The details of these equations do not concern us here. The important point is that they all have the form

$$\left\langle \tilde{\vec{\Gamma}}(\vec{r},\vec{r}\,';s)\ ;\ \tilde{\vec{J}}(\vec{r}\,',s)\right\rangle = \tilde{\vec{I}}(\vec{r},s)$$

$$\begin{aligned} \tilde{\vec{I}}(\vec{r},s) &\equiv \text{incident or source field of some kind (specified)} \\ \tilde{\vec{\Gamma}}(\vec{r},\vec{r}\,';s) &\equiv \text{kernel (related to Green's function) which may be a distribution} \\ \tilde{\vec{J}}(\vec{r}\,',s) &\equiv \text{typically current density or surface current density} \end{aligned} \tag{1.2}$$

Here

$$\langle\, ,\rangle \equiv \text{symmetric product} \tag{1.3}$$

is our convenient way to indicate multiplication (of the two terms separated by the comma) followed by integration with respect to the common spatial coordinates over the domain of the scatterer; the type of multiplication (e.g., dot ($\cdot$) or cross ($\times$) product) is indicated by appropriate symbols above the comma. With additional commas this symmetric product is extended to as many terms and integrations as desired.

One can in principle solve the integral equation by inverting the integral operator. One formally determines an inverse kernel (which may be a distribution) which gives a solution

$$\begin{aligned} \tilde{\vec{J}}(\vec{r},s) &= \left\langle \tilde{\vec{\Gamma}}^{-1}(\vec{r},\vec{r}\,';s)\ ;\ \tilde{\vec{I}}(\vec{r}\,',s)\right\rangle \\ \left\langle \tilde{\vec{\Gamma}}(\vec{r},\vec{r}\,'')\ ;\ \tilde{\vec{\Gamma}}^{-1}(\vec{r}\,'',\vec{r}\,';s)\right\rangle &= \overleftrightarrow{1}(\vec{r}-\vec{r}\,') \equiv \text{identity on scatterer} \end{aligned} \tag{1.4}$$

where the identity is taken in the sense of the relevant vector components and domain of integration (e.g., two or three dimensions for surfaces or volumes, respectively).

For SEM these integral equations have proven to be very useful in constructing formulae for the various terms. Singularity expansions can be constructed for both the response $\tilde{\vec{J}}$ and the inverse kernel $\tilde{\vec{\Gamma}}^{-1}$ (related to the class 1 and class 2 forms of the coupling coefficient, respectively). Furthermore, the integral-equation kernels can be used to construct eigenmode expansions which give additional insight into the SEM terms.

1.6 Matrix and Operator Theory

Integral equations have been cast in approximate numerical form by the moment method (MoM). In this numerical solution procedure (typically for use with large digital computers) the current density (response) is expanded in a set of functions (of finite number in practice) called expansion functions; the incident or source field is similarly expanded in a set of testing functions.* The vectors of coefficients of these two sets (taken with equal numbers of components) are related by a matrix (square) which replaces the integral-equation operator in the form

$$(\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{J}_n(s)) = (\tilde{I}_n(s)) \tag{1.5}$$

Inverting the matrix, one has an approximate solution to the original equation (1.2) in a form analogous to (1.4). In matrix form our equation is more familiar to electrical engineers because such types of equations appear in circuit problems. The arsenal of matrix theory is now at our disposal. Eigenvectors and eigenvalues can be constructed for representing the solution and understanding its properties. Combining matrix (or operator) theory with complex variable theory is essential to SEM. This paper will not delve into the mathematical theory of such operators, this subject being left to others.

2. EARLY DEVELOPMENT OF SEM

2.1 The Beginning

In early 1971 the question was posed (by this author). Experimental observations of damped sinusoids in EMP experiments† suggested poles in the corresponding Laplace transforms. Then in Laplace-transform or complex-frequency domain this led to the idea of expanding the response in terms of all the singularities *in the complex frequency plane*. Besides poles, such singularities might include branch points and associated integrals, essential singularities, and (for completeness) entire function(s) corresponding to any singularities at infinity.

Concentrating on the poles it was observed that, except for poles in the exciting waveform (transformed), these were the natural frequencies of the scatterer or antenna because integral equations describing the object response would admit non-trivial responses at such frequencies with no excitation. Said another way, the response at an object pole is infinite if the excitation is non-zero at such a complex frequency. Interpreting (1.2) in this sense gives

$$\begin{gathered}\left\langle \tilde{\vec{\Gamma}}(\vec{r},\vec{r}\,';s_\alpha)\ ;\ \vec{J}_\alpha(\vec{r}\,')\right\rangle = \vec{0} \quad , \quad s_\alpha \equiv \text{natural frequency} \\ \vec{J}_\alpha(\vec{r}) \equiv \text{natural mode corresponding to } s_\alpha\end{gathered} \tag{2.1}$$

or from (1.5) in MoM form

$$(\tilde{\Gamma}_n(s_\alpha)) \cdot (j_n)_\alpha = (0_n) \quad , \quad \det((\tilde{\Gamma}_n(s_\alpha)) = 0 \tag{2.2}$$

which gives a way of computing natural frequencies. Noting that the matrix is singular (and hence so is its transpose) we can write

$$(\mu_n)_\alpha \cdot (\tilde{\Gamma}_n(s_\alpha)) = (0_n) \quad , \quad \left\langle \vec{\mu}_\alpha(\vec{r})\ ;\ \tilde{\vec{\Gamma}}(\vec{r},\vec{r}\,';s_\alpha)\right\rangle = \vec{0} \tag{2.3}$$

*Harrington, R. F., Field Computation by Moment Methods, Macmillan, 1968.

†Joint Special Issue on the Nuclear Electromagnetic Pulse, IEEE Trans. Antennas and Propagation, AP-26, Jan 1978, and IEEE Trans. EMC, EMC-20, Feb 1978.

$\vec{\mu}_\alpha(\vec{r}) \equiv$ coupling mode corresponding to s_α

where the use of the coupling mode will become clear later. For the common case of a symmetric kernel (as in the E-field or impedance integral equation) the coupling mode can be set equal to the natural mode. The choice of a normalization for these modes is somewhat arbitrary.

Having equations for the natural frequencies and modes then construct a solution in the form

$$\tilde{\vec{U}}(\vec{r},s) = \sum_\alpha \tilde{\eta}_\alpha \vec{J}_\alpha(\vec{r})(s - s_\alpha)^{-1} + \text{other singularity terms}$$

$$\equiv \text{normalized (delta-function) response to incident or source field } E_0^{-1}\tilde{f}^{-1}(s)\tilde{\vec{I}}(\vec{r},s)$$

$$\tilde{\vec{J}}(\vec{r},s) = E_0\tilde{f}(s)\tilde{\vec{U}}(\vec{r},s) \quad , \quad \tilde{\eta}_\alpha \equiv \text{coupling coefficient} \tag{2.4}$$

$\tilde{f}(s) \equiv$ incident or source waveform (Laplace transformed)

$E_0 \equiv$ scaling amplitude for incident waveform

where the postulated coupling coefficient contains the spatial characteristics of the incident field. Here first order poles have been assumed, although higher order poles can be included. One can also include the incident waveform in the pole residues as

$$\tilde{\vec{J}}(\vec{r},s) = E_0 \sum_\alpha \tilde{f}(s_\alpha)\tilde{\eta}_\alpha \vec{J}_\alpha(\vec{r})(s - s_\alpha)^{-1} + \text{other singularity terms} \tag{2.5}$$

This was the general state of knowledge on this subject when in September 1971 a special meeting was held at Northrop Corporate Laboratories office in Pasadena, California. Many prominent electromagnetic specialists participated in this discussion of SEM. The basic concepts were presented as outlined above to stimulate basic ideas and potential application to areas such as EMP data analysis, target identification, equivalent circuits, etc.

2.2 Evaluation of the Coupling Coefficient

In late 1971 a key discovery was made in that formulae for the coupling coefficient were developed in terms of the integral-equation terms in (1.2). This was done independently with different approaches by Baum [3.1] and by Marin and Latham [3.7]. The details of these derivations need not concern us here as they were rather involved. Subsequent papers have simplified this somewhat.

Noting that the kernel $\tilde{\overleftrightarrow{\Gamma}}$ and normalized incident or source field

$$\tilde{\vec{I}}^{(n)}(\vec{r},s) = E_0^{-1}\tilde{f}^{-1}(s)\tilde{\vec{I}}(\vec{r},s) \tag{2.6}$$

are analytic functions of s near s_α, expand them in a power series in $s - s_\alpha$. Collecting terms and applying the coupling vector leads to the class 1 coupling coefficient

$$\tilde{\eta}_\alpha^{(1)} = e^{-(s-s_\alpha)t_0} \frac{\langle \vec{\mu}_\alpha(\vec{r}) \; ; \; \tilde{\vec{I}}^{(n)}(\vec{r},s_\alpha)\rangle}{\langle \vec{\mu}_\alpha(\vec{r}) \; ; \; \frac{d}{ds}\tilde{\overleftrightarrow{\Gamma}}(\vec{r},\vec{r}\,';s)\big|_{s=s_\alpha} \; ; \; \vec{j}_\alpha(\vec{r}\,')\rangle} \tag{2.7}$$

where the turn-on time t_0 can be a function of the observer position $\vec{r}$. An alternate form is the class 2 coupling coefficient which results from first finding the SEM representation (strict) of $\overleftrightarrow{\Gamma}^{-1}$ and then as in (1.4) operating on $\tilde{\vec{I}}^{(n)}(\vec{r}\,',s)$ with the poles of $\overleftrightarrow{\Gamma}^{-1}$ giving

$$\tilde{\eta}_\alpha^{(2)} = \frac{\langle e^{-(s-s_\alpha)t_0}\vec{\mu}_\alpha(\vec{r}) \; ; \; \tilde{\vec{I}}^{(n)}(\vec{r},s)\rangle}{\langle \vec{\mu}_\alpha(\vec{r}) \; ; \; \frac{d}{ds}\tilde{\overleftrightarrow{\Gamma}}(\vec{r},\vec{r}\,';s)\big|_{s=s_\alpha} \; ; \; \vec{j}_\alpha(\vec{r}\,')\rangle} \tag{2.8}$$

where the turn-on time t_0 can here be a function of both $\vec{r}$ and $\vec{r}\,'$. See [2.1] for a more complete derivation.

The two classes of coupling coefficients have some significant differences. Except for a delay factor the class 1 form is particularly simple, being independent of s, so that in time domain the normalized response in (2.4) takes the form

$$\vec{U}(\vec{r},t) = \sum_\alpha \tilde{\eta}_\alpha^{(0)}\vec{j}_\alpha(\vec{r}) \, e^{s_\alpha t} \, u(t - t_0) + \text{other singularity terms} \tag{2.9}$$

Here the coupling coefficient at $s = s_\alpha$ is

$$\tilde{\eta}_\alpha^{(0)} = \tilde{\eta}_\alpha^{(1)}\big|_{s=s_\alpha} = \tilde{\eta}_\alpha^{(2)}\big|_{s=s_\alpha} = \frac{\langle \vec{\mu}_\alpha(\vec{r}) \; ; \; \tilde{\vec{I}}^{(n)}(\vec{r},s_\alpha)\rangle}{\langle \vec{\mu}_\alpha(\vec{r}) \; ; \; \frac{d}{ds}\tilde{\overleftrightarrow{\Gamma}}(\vec{r},\vec{r}\,';s)\big|_{s=s_\alpha} \; ; \; \vec{j}_\alpha(\vec{r}\,')\rangle} \tag{2.10}$$

so that both classes reduce to the same thing at the pole ($s = s_\alpha$). While the class 1 form gives simple damped sinusoids the class 2 form gives a convolution as

$$\vec{U}(\vec{r},t) = \sum_\alpha \vec{j}_\alpha(\vec{r})\eta_\alpha^{(2)} \circ [e^{s_\alpha t} \, u(t)] + \text{other singularity terms}$$

$$\circ \equiv \text{convolution with respect to time} \tag{2.11}$$

$$\eta_\alpha^{(2)} = \frac{\langle \vec{\mu}_\alpha(\vec{r})[e^{s_\alpha t_0} \, u(t - t_0)] \overset{\circ}{;} \vec{I}^{(n)}(\vec{r},t)\rangle}{\langle \vec{\mu}_\alpha(\vec{r}) \; ; \; \frac{\partial}{\partial s}\tilde{\overleftrightarrow{\Gamma}}(\vec{r},\vec{r}\,';s)\big|_{s=s_\alpha} \; ; \; \vec{j}_\alpha(\vec{r}\,')\rangle}$$

At late times the time-domain pole terms in (2.9) and (2.11) give the same simple damped sinusoids. For $t_0 = 0$ in class 2, and t_0 (typically used) in class 1 chosen on or before the wave reaches the scatterer, class 1 and class 2 give identical pole terms after the wave passes the body. There are numerous details concerning the properties of the two classes omitted here. A recent paper goes into this topic in greater depth [3.5].

2.3 Example Problems

Now that the floodgates were open numerous investigators considered specific finite-size scatterers in free space. The early examples were the sphere (analytically) [3.1], the thin wire (approximate) [4.26], and the thin wire by numerical (MoM) computation [4.48]. The reader can consult the bibliography in this special issue for many more examples. A review book chapter by this author [2.1] summarizes most of the early examples of this type.

3. LATER DEVELOPMENTS

3.1 Natural Modes for Radiated or Scattered Fields

An early extension of the SEM concepts was to go from the currents and charges on an object to the radiated or scattered fields in the space surrounding the object. In 1973 there were papers by Tesche [4.49] concerning the numerical calculation of the far fields from linear antennas in terms of natural modes, and by Baum [3.3] concerning the formalism of such natural modes for near and far fields. These results established a concept of transient antenna (or scatterer) patterns in terms of natural frequencies, modes, and coupling coefficients.

3.2 Analysis of Experimental Data

Since the original impetus toward SEM came from observations of the general properties of the transient electromagnetic response of systems, it is understandable that the general SEM theory should be applied to such experimental data. Certain SEM parameters are in principle experimentally observable. In 1974 a paper (USNC/URSI meeting, Boulder, Colorado, October 1974, later in [5.16]) by VanBlaricum and Mittra applied the Prony technique to transient EM scattering waveforms to find the natural frequencies and residues by fitting the waveform with a sum of damped sinusoids. Since then many investigators have tried various other techniques in attempts to increase speed of computation, minimize the effect of noise in the waveform, and maximize the accuracy in determining the true poles in the scattering data.

3.3 Eigenmode Expansion Method (EEM)

In 1975 this author introduced the eigenmode expansion method to find more properties of the SEM [3.4]. One defines eigenvalues and eigenmodes for the integral operator (kernel) in (1.2) via

$$\begin{aligned} &\left\langle \tilde{\vec{\Gamma}}(\vec{r},\vec{r}\,';s)\ ;\ \tilde{\vec{j}}_\beta(\vec{r}\,',s)\right\rangle = \tilde{\lambda}_\beta(s)\tilde{\vec{j}}_\beta(\vec{r},s) \\ &\left\langle \tilde{\vec{\mu}}_\beta(\vec{r},s)\ ;\ \tilde{\vec{\Gamma}}(\vec{r},\vec{r}\,';s)\right\rangle = \tilde{\lambda}_\beta(s)\tilde{\vec{\mu}}_\beta(\vec{r}\,',s) \\ &\tilde{\lambda}_\beta(s) = \text{eigenvalue} \\ &\tilde{\vec{j}}_\beta(\vec{r},s) = \text{right eigenmode}\quad,\quad \tilde{\vec{\mu}}_\beta(\vec{r},s) = \text{left eigenmode} \end{aligned} \tag{3.1}$$

Unlike the natural modes the eigenmodes can be generally biorthonormalized as

$$\left\langle \tilde{\vec{\mu}}_{\beta_1}(\vec{r},s)\ ;\ \tilde{\vec{j}}_{\beta_2}(\vec{r},s)\right\rangle = 1_{\beta_1,\beta_2} = \begin{cases} 1 \text{ for } \beta_1 = \beta_2 \\ 0 \text{ otherwise} \end{cases} \tag{3.2}$$

giving the representations for the kernel (and its inverse)

$$\tilde{\overleftrightarrow{\Gamma}}^{n}(\vec{r},\vec{r}\,';s) = \sum_{\beta} \tilde{\lambda}_{\beta}^{n}(s)\tilde{\vec{j}}_{\beta}(\vec{r},s)\tilde{\vec{\mu}}_{\beta}(\vec{r}\,',s) \tag{3.3}$$

and the response

$$\tilde{\vec{U}}(\vec{r},s) = \sum_{\beta} \tilde{\lambda}_{\beta}^{-1}(s) \left\langle \tilde{\vec{\mu}}_{\beta}(\vec{r}\,',s) \; ; \; \tilde{\vec{I}}^{(n)}(\vec{r}\,',s) \right\rangle \tilde{\vec{j}}_{\beta}(\vec{r},s) \tag{3.4}$$

While there are various mathematical problems to be considered concerning completeness, root vectors, sense of convergence, etc., there are some approximate ways to view this matter. Casting the integral equation (1.2) into matrix (MoM) numerical form as in (1.5), the EEM is considered as a problem of finding the eigenvalues and left and right eigenvectors of $(\tilde{\Gamma}_{n,m}(s))$.

Summarizing some of the SEM related results we have

$$\tilde{\lambda}_{\beta}(s_{\beta,\beta'}) = 0 \quad , \quad s_{\beta,\beta'} \equiv s_{\alpha} \tag{3.5}$$

so that the natural frequencies are zeros of particular eigenvalues (hence $\alpha \rightarrow (\beta,\beta')$), so that the eigenvalues order or partition the set of natural frequencies. Similarly for the modes (with appropriate normalizations)

$$\tilde{\vec{j}}_{\beta}(\vec{r},s_{\beta,\beta'}) = \vec{j}_{\beta,\beta'}(\vec{r}) \quad , \quad \tilde{\vec{\mu}}_{\beta}(\vec{r},s_{\beta,\beta'}) = \vec{\mu}_{\beta,\beta'}(\vec{r}) \tag{3.6}$$

For the denominator in the coupling coefficients we have

$$\left\langle \vec{\mu}_{\alpha}(\vec{r}) \; ; \; \frac{d}{ds}\tilde{\overleftrightarrow{\Gamma}}(\vec{r},\vec{r}\,';s)\Big|_{s=s_{\alpha}} \; ; \; \vec{j}_{\alpha}(\vec{r}\,') \right\rangle = \frac{d}{ds}\,\tilde{\lambda}_{\beta}(s)\Big|_{s=s_{\beta,\beta'}} \tag{3.7}$$

which allows us to represent class 1 (in (2.7)) and class 2 (in 2.8)) in terms of EEM quantities.

Another application of EEM is to the synthesis of transient responses via changing the eigenvalues. Eigenimpedance synthesis considers the eigenvalues $\tilde{Z}_{\beta}(s)$ of the impedance (or E-field) integral equation and notes that, if the scatterer or antenna is impedance loaded in certain ways $(\tilde{Z}_{\ell}(s))$, the eigenimpedances are modified as

$$\tilde{Z}_{\beta}(s) \rightarrow \tilde{Z}_{\beta}(s) + \tilde{Z}_{\ell}(s) \tag{3.8}$$

which allows one to synthesize a $\tilde{Z}_{\ell}(s)$ to move the natural frequencies $s_{\beta,\beta'}$ to other more desirable positions in the complex s plane. These EEM matters are necessarily quite abbreviated here. More complete reviews are included in [2.2,2.3]. Of special note is the recent extension of Sancer et al. [3.11] in which the eigenmodes of the "pseudosymmetric" H-field integral equation are paired with corresponding eigenvalues (normalized) adding to 1.0.

3.4 Target Identification

In the original development of the SEM concept (section 2.1) it was noted that the natural frequencies of a scatterer were independent of the exciting fields. This was considered a potentially useful property for target identification purposes. In 1975 two groups published papers proposing techniques for this general kind of target identification,* based on work dating from about 1974. Another group† gave a spoken paper on this subject in 1975 also. This was also about the time (1975) of the introduction of the concept of eigenimpedance synthesis for modifying the pole pattern in the s plane to make the identification more difficult [3.4].

3.5 Equivalent Circuits for Antennas and Scatterers

In 1976 this author showed how to construct formal equivalent circuits at an antenna/scatterer port from the SEM representation [4.3]. A review of this development is included in [2.3]. The key to this development is to note that the admittance and short-circuit current (or the impedance and open-circuit voltage) have the same pole locations in the s plane because they have the same integral-equation operator; only the source fields are different. For the short-circuit boundary value problem this leads to a parallel combination of series "resonant" circuits with series voltage sources. For the open-circuit boundary value problem one has the dual situation of a series combination of parallel "resonant" circuits with parallel current sources. More recent investigations have centered on canonical problems for exploring the realizability of such networks. Results have been obtained by Pearson et al. [4.33, 4.34, 4.39], Singaraju and Baum [4.2], and Sharpe and Roussi [4.44a].

3.6 Calculation of Natural Frequencies

Initial computations of the natural frequencies from the MoM matrix determinant in (2.2) were by classical Newton and Muller zero-searching techniques [5.3]. Following an early paper in 1974 [5.1], Baum, Giri, and Singaraju developed contour integral techniques including computer programs to efficiently and accurately compute all the natural frequencies in a given portion of the s plane [5.15,5.7]. This is also reviewed in [2.3]. Also of interest is the variational technique based on EEM concepts proposed by Mittra and Pearson [5.10].

3.7 Fora and Reviews

An important milestone in SEM development was the first special session at a USNC/URSI meeting in Boulder, Colorado, August 1973. Since that time there have been many SEM sessions at the various USNC/URSI meetings and IEEE Antennas and Propagation symposia. Reviews on the subject have been given at the triennial URSI General Assemblies beginning with the one in Lima, Peru, in 1975. This author has written three major review papers and book chapters on this subject [2.1-2.3]; these can be consulted for more complete developments and numerous references. A review [2.4] by Dolph and Scott treats some of the applicable mathematical theory. Now SEM has reached another milestone with the

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*Pearson, L.W., M.L. VanBlaricum, and R. Mittra, A New Method for Radar Target Recognition Based on the Singularity Expansion Method, Record of IEEE International Radar Conference, Arlington, Virginia, April 1975, pp. 452-457.

Moffatt, D.L., and R.K. Mains, Detection and Discrimination of Radar Targets, IEEE Trans. Antennas and Propagation, May 1975, pp. 358-367.

†Deadrick, F.J., H.G. Hudson, E.K. Miller, J.A. Landt, and A.J. Poggio, Object Identification via Pole Extraction from Transient Fields, USNC/URSI Meeting, U. of Illinois, 3-5 June 1975, p. 67.

recent symposium: "Mathematical Foundations of the Singularity Expansion Method," University of Kentucky, November 1980. This special SEM issue is the proceedings of that symposium.

4. CONTINUED DEVELOPMENT

Quo vadimus? Quo vadit SEM? These are difficult questions. SEM is currently being pursued on two levels. First there is the engineering theory and applications oriented to meeting the practical needs of transient and broadband EM applications such as EMP, lightning, and target identification. This is even finding application in acoustic target identification (see Überall and Gaunard references, this issue). It is these applications oriented developments that I have concentrated on in this paper. On another level the mathematicians are pursuing a rigorous exploration of the SEM theory with a view to defining the precise limits of applicability. Other papers in this issue address such points.

From an applications point of view I see some important areas, both theoretical and experimental, for future development. For experimental description of complex electronic equipment we need to apply all our powerful insights concerning the SEM description to obtaining <u>all</u> the SEM pole (and other) parameters from the experimental scattering (or interaction) data. Using (2.5) (in frequency and/or time domains) one can use the factoring of the pole terms to exhibit the dependence of the response on the various separate parameters of the scattering problem. This gives a much more compact representation of the data (in the resonant region) allowing one to much more readily see the important features, including worst cases, etc. of the response. This factorization can also likely be used to more accurately evaluate the SEM parameters by having (2.5) simultaneously fit many data records corresponding to different locations and excitation conditions.

The construction of equivalent circuits needs much more development. Alternate canonical forms (such as ladder networks, etc.) need to be developed. Perhaps other expansions such as a low-frequency expansion [2.2] could be useful in conjunction with SEM and EEM. Both a deeper understanding of SEM/EEM decomposition of scatterer response, and more accurate and efficient obtaining of these parameters from experimental data, are needed for the target identification problem. This area has a very great practical potential.

MAJOR RESULTS AND UNRESOLVED ISSUES IN SINGULARITY EXPANSION METHOD

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ABSTRACT

The Singularity Expansion Method (SEM) was derived as a means of interpreting/estimating responses measured during electromagnetic testing of aerospace systems. These responses appeared to consist of a superposition of exponentially damped sinusoidal oscillations. It was shown that the electromagnetic response of a finite-sized, perfectly conducting object to a delta-function incident plane wave is a meromorphic function of the complex frequency. The physical interpretation, computational advantages and fundamental problems associated with using the poles (natural frequencies) of the meromorphic function to construct the transient responses of objects are reviewed. Areas of future investigations, both for the purpose of improving the mathematical foundations and the computational tools are discussed.

1. INTRODUCTION

Small-sized electronic circuits, whether they use discrete components or integrated circuits, are susceptible to malfunction or damage caused by transient interference. The problems are particularly common in data processing circuits because these circuits often cannot distinguish between a spurious transient and a legitimate signal and because these circuits are designed for small switching levels to conserve power and reduce heat dissipation problems. Logic levels are often a few volts or a few tens of milliamperes in these circuits.

On the other hand, transients associated with EMP, lightning and switching on buried communication cables can have peak values of tens of kiloamperes. In order to protect electronic equipment onboard these systems it is necessary to understand how electromagnetic waves couple into the systems. When treating these problems it is advantageous to divide them into three different parts, namely

- external interaction problems
- penetration problems
- internal interaction problems

The external interaction problems consist of finding the surface current and charge densities induced by an incident electromagnetic wave on the

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0272-6343/81/040361-13$2.25

exterior surface of the considered system (aircraft, satellite, missile, etc.). The penetration problems consist of determining how electromagnetic energy penetrates through the exterior surface of the system. Mechanisms that play a role in the penetration are (1) diffusion through metal skins, (2) field leakage through nonconducting portions of the system surface, (3) signals on lines passing through the surface. Finally, the internal interaction problems consist of estimating electromagnetic quantities inside the surface such as currents and voltages on wires and cables and fields in various cavities. The solution of the external coupling problem serves as an input to the penetration problem. The quantities obtained from the penetration problem then form the sources for the internal coupling problem. The singularity expansion method was developed primarily as a mathematical tool for attacking the external interaction problem.

2. THE EXTERNAL INTERACTION PROBLEM

2.1 Experimental Results

Figure 1 shows some typical responses obtained during tests of aerospace systems. The curve in Figure 1a refers to the current induced on a wire inside an aircraft when the aircraft is exposed to a pulsed electromagnetic field. The late-time behavior of the curve consists of a damped sinusoidal oscillation. The curve in Figure 1b shows the current density

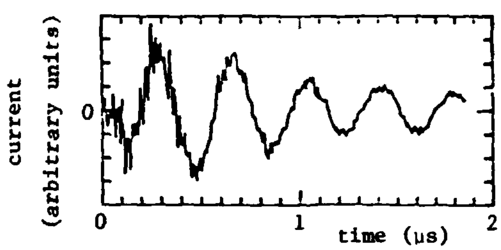

Figure 1a. Typical result from test of aerospace system.

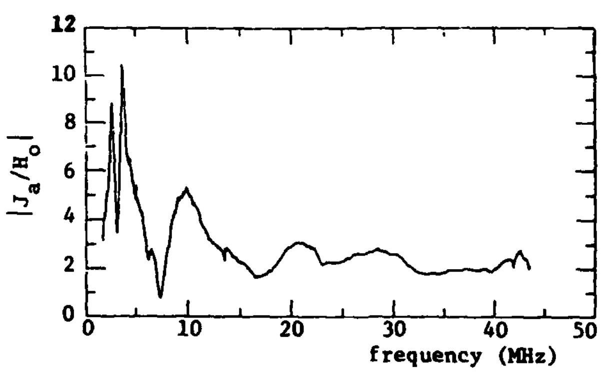

Figure 1b. Results of measurements on scale model of Boeing 707.

on the skin of a scale-model aircraft when it is exposed to a swept CW plane wave. The fundamental resonance in Figure 1b corresponds to the frequency of the damped sinusoidal oscillation in Figure 1a. The many results similar to those shown in Figure 1 obtained during EM testing of aerospace systems indicate that the system response can be constructed from its natural (or eigen) frequencies in much the same way as in network theory. In mathematical terms this translates into showing that the response function can be constructed from a meromorphic (in the complex frequency plane) operator operating on the incident field.

2.2 Mathematical Proof Regarding Meromorphicity of External Response

The original proof showing that the response of a finite-sized perfectly conducting object indeed can be constructed using a meromorphic operator is shown in [3.9]. In this proof the analytical properties in the complex frequency (s) plane of the surface current density $\underline{j}$ induced on a finite-sized perfectly conducting object is investigated using the magnetic-field integral equation. This equation can be cast in the following form:

$$\left[\frac{1}{2}\,\underline{\underline{I}} - \underline{\underline{L}}(s)\right]\cdot\underline{j} = \underline{n}\times\underline{H}^{inc}, \qquad \underline{\underline{L}}\cdot\underline{j} \equiv \int_S \underline{n}\times(\nabla G\times\underline{j})\,dS' \tag{1}$$

$G(\underline{r},\underline{r}';s) = (4\pi|\underline{r}-\underline{r}'|)^{-1}\exp(-s|\underline{r}-\underline{r}'|/c)$, and the surface S is finite. From the Fredholm theory for the solution of integral equations of the second kind it is shown that the inverse operator $(\frac{1}{2}\,\underline{\underline{I}}-\underline{\underline{L}})^{-1}$ is a meromorphic operator-valued function of s. The locations of the poles of the inverse operator (the natural frequencies) are given by those values of $s(s_n)$ for which the homogeneous integral equation has a nontrival solution $\underline{j}_n$,

$$\left[\frac{1}{2}\,\underline{\underline{I}} - \underline{\underline{L}}(s_n)\right]\cdot\underline{j}_n = 0, \qquad \left[\frac{1}{2}\,\underline{\underline{I}} - \underline{\underline{L}}^{\dagger}(s_n)\right]\cdot\underline{h}_n = 0 \tag{2}$$

and $\underline{\underline{L}}^{\dagger}$ is the adjoint operator of $\underline{\underline{L}}$,

$$\underline{\underline{L}}^{\dagger}\cdot h = -\int_S \nabla G^{*}\times(\underline{n}\times\underline{h})\,dS'$$

Since poles are the only singularities in the s-plane of the inverse operator the Mittag-Leffler theorem can be invoked to find an explicit representation of this inverse operator in terms of the natural frequencies, the nontrivial solutions of the homogeneous integral equation and the nontrivial solutions of the homogeneous adjoint integral equation,

$$\left[\frac{1}{2}\,\underline{\underline{I}} - \underline{\underline{L}}(s)\right]^{-1} = \sum_n \left\{(s-s_n)^{-1}\left[\langle\underline{\underline{B}}_n\cdot\underline{j}_n,\ \underline{h}_n\rangle\right]^{-1}\times\right.$$
$$\left.\times\ \underline{j}_n\underline{h}_n^{*} + \underline{\underline{P}}_n(s)\right\} + \underline{\underline{E}}(s) \tag{3}$$

where $\underline{\underline{B}}_n = (d\underline{\underline{L}}_n/ds)(s_n)$, $\underline{\underline{P}}_n(s)$ are polynomial operator-valued functions of s, and $\underline{\underline{E}}(s)$ is an entire operator valued function of s.*

*E.Goursat, Functions of a Complex Variable, Dover Publications, Inc., N.Y. 1959

In deriving (3) it has been assumed that all poles are simple poles and that there are only a finite number of poles in every finite portion of the complex frequency plane. This assumption has been substantiated numerically for all perfectly conducting finite bodies investigated so far. In the case of objects satisfying impedance boundary conditions poles of higher order have been found.*

The proof in [3.9]is obtained by seeking solutions in the Hibert space of elements $\underline{j}(\underline{r})$, $\underline{r} \in S$ which are tangent to S, and then applying the method of Carelman to an equation derived from (1) whose kernel is shown to be of the Fredholm type. A simpler and more general proof can be obtained by using the analytic Fredholm theory derived in [6.95] and [6.22].

2.3 Strength and Incompleteness of Derived Expression

From the representation (3) of the inverse operator it is observed that

- poles correspond to natural (free) oscillations of the scattering object,
- locations of poles depend solely on shape and size of scattering object,
- each natural oscillation has its associated current distribution (that is the nontrivial solution of the homogeneous integral equation).

These observations show that many transient scattering and antenna problems involving finite-sized objects can be treated by employing the same methods as those used in transient network and transmission-line theory.

The expression (3) also points to some of the unresolved questions in SEM. Mathematically, the question can be formulated: "How are the polynomial operators $\underline{\underline{P}}_n(s)$ (that are introduced so that the Mittag-Leffler series converges) and the entire operator $\underline{\underline{E}}(s)$ determined?" In some cases, such as scattering from a sphere and from thin wires, a series can be constructed where these operators are explicitly determined. To make full use of the SEM it is necessary that this issue be completely resolved. However, even without a complete knowledge of the $\underline{\underline{P}}_n(s)$ and $\underline{\underline{E}}(s)$ the series expansion (3) can be used to construct a time domain representation of the transient response valid in a certain time regime.

2.4 An Expression for Transient Response

The transient response can be obtained from the frequency domain expression (3) by way of an inverse Laplace transform. This integral can be evaluated as a sum of residues of the poles at s_n plus any contributions from singularities in the incident field provided that one can close the integration path at infinity. This leads one to investigate the inverse operator $(\frac{1}{2}\underline{\underline{I}} - \underline{\underline{L}})^{-1}$ for large values of $|s|$.

One can show using standard techniques that

$$\left(\frac{1}{2}\,\underline{\underline{I}} - \underline{\underline{L}}\right)^{-1} \to 2\underline{\underline{I}} \quad \text{as } \mathrm{Re}\{s\} \to +\infty \tag{4}$$

*C.T.Tai,Complex Singularity of the Impedance Functions of Antennas,this issue.

To obtain the behavior of the inverse operator as $|s| \to \infty$, in general, requires more elaborate analysis. Such an analysis is carried out by Jones* using the theory of entire functions in complex variable theory. From this theory it follows that

$$\left\| \left(\frac{1}{2} \underline{\underline{I}} - \underline{\underline{L}} \right)^{-1} \right\| < \exp[(d+\eta)|s|/c] \quad \text{as} \quad |s| \to \infty \tag{5}$$

for any $\eta > 0$. The quantity d is the maximum separation between any pair of points on S (sometimes referred to as the diameter of the object). The result (5) can be interpreted to mean that the resolvent kernel $\underline{\underline{R}}(\underline{r},\underline{r}',s)$ to the kernel in the magnetic field integral equation is a meromorphic function of order one.

Let the incident field be a δ-function plane wave such that $\underline{H}^{inc}(\underline{r},t) = \underline{I}_o \delta(t - s\underline{e}\cdot\underline{r}/c)$ where $\underline{e}$ is the direction of propagation of the wave. One can then derive the following expression for the time history of the induced surface current $\underline{J}(\underline{r},t)$

$$\underline{J}(\underline{r},t) = 2\underline{n}\times\underline{H}^{inc} + \frac{1}{2\pi i}\int_{Br} \exp(st)ds \int_S \underline{\underline{R}}(\underline{r},\underline{r}',s)\cdot\Big[(\underline{n}'\times\underline{I}_o)\times$$
$$\times \exp(-s\underline{e}\cdot\underline{r}'/c)\Big]dS' \tag{6}$$

The coordinate systems are chosen such that the object is directly illuminated between t_1 and t_2.

Some results can be deduced from (4) and (6). They are

- $\underline{J}(\underline{r},t) = 0, \quad t < t_1$ which is in accordance with the causality condition
- The domain of integration S in (6) can be reduced to $S' = \{\underline{r}' : \underline{r}' \in S \text{ and } |\underline{r}-\underline{r}'| + \underline{e}\cdot\underline{r}' < ct\}$

The limitation in the growth of $\underline{\underline{R}}$ deduced in (5) can be used to close the Bromwich contour in (6) in the left half plane resulting in the following expression

$$\underline{J}(\underline{r},t) = \sum_n \exp(s_n t)\left[\langle B_n\cdot\underline{j}_n, \underline{h}_n\rangle\right]^{-1}\underline{j}_n(\underline{r})$$
$$\int_S (\underline{n}'\times\underline{I}_o)\cdot\underline{h}_n(\underline{r}')\exp(-s_n\underline{e}\cdot\underline{r}'/c)dS' \tag{7}$$

which is valid for $t > t_2 + d/c$. The result (7) shows that the SEM expansion converges within a given late time regime. It still remains to derive an SEM representation that is valid for all times.

3. CALCULATION OF THE RESPONSE OF DIFFERENT OBJECTS

The results exhibited in equation (3) can be used to determine the response of various objects. The quantities s_n and $\underline{j}_n$ can be interpreted as the resonance frequency and current distribution of a natural mode.

*D.S. Jones, Methods in Electromagnetic Wave Propagation, Oxford, 1979.

Having established the existence of this mode one can use different methods of calculating pertinent quantities of the mode. A few cases will be treated in this paper, namely,

- bodies of revolution,
- simple sticks,
- stick-model aircraft.

3.1 Numerical Determination of Natural Modes and Transient Response of Rotationally Symmetric Objects

For a rotationally symmetric object as shown in Figure 2 the surface current density $\underline{j}$ can be expanded in a Fourier series in the azimuthal angle ϕ. One dimensional integral equations with the arc-length coordinate ξ as the independent variable can then be formulated for each Fourier component of the current.

This method was used to numerically determine the resonance frequencies and current distributions of the lowest natural modes of a prolate spheroid.

In Figure 3 we graph the locus of some of the natural frequencies as the length of the minor axis (2a) varies, but the length of the major axis (2b) is fixed. The quantity d in Figure 3 is one quarter of the circumference of an ellipse with semimajor axis b and semiminor axis a. For poles close to the imaginary axis we note that the absolute value of the real part of sd/c stays almost constant. This means that the Q value of each mode is a decreasing function of a when b is fixed. For the other poles we note that the absolute value of the real part of sd/c is a decreasing function of a.

The current distribution of some natural modes is depicted in Figure 4. The current density, j_{ℓ_n}, is so normalized that its absolute value is less

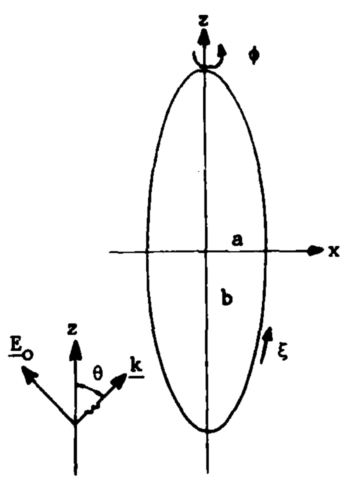

Figure 2. Plane wave impinging on prolate spheroid.

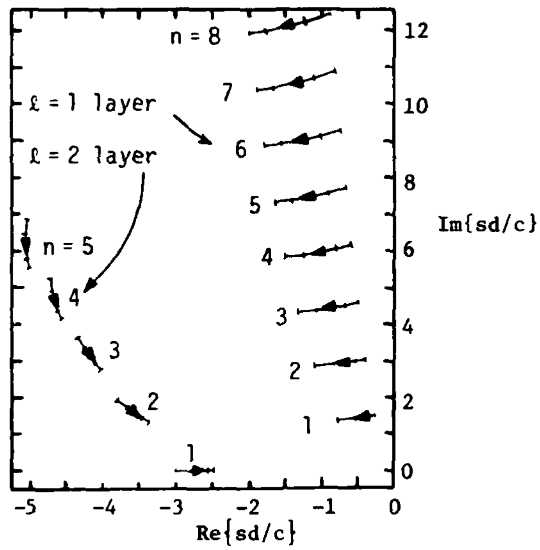

Figure 3. Loci of natural frequencies when $0.1 \leq a/b \leq 1$. The locations of the natural frequencies for a/b = 0.1, 0.2, 0.5, and 1 are indicated on the curves. Arrow indicates direction in which natural frequencies vary for increasing values of a/b.

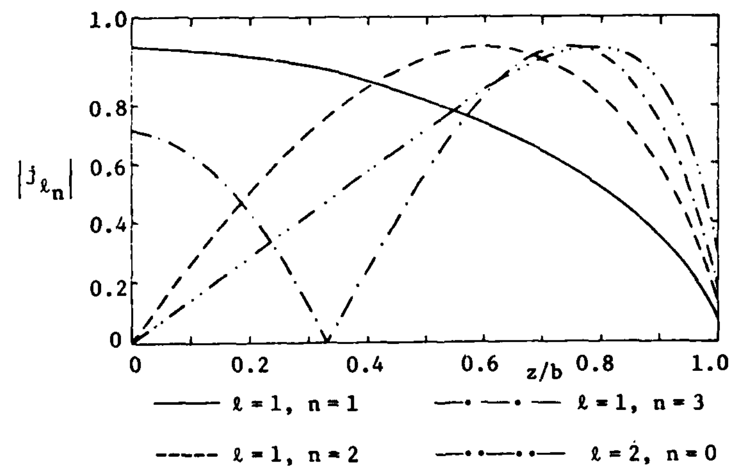

Figure 4. Current distribution of natural modes of prolate spheroid where a/b = 0.1.

than or equal to one. The current distribution is an odd (even) function with respect to the xy plane for modes where the index n is an even (odd) integer. Moreover, the current distribution is a real function for modes whose natural frequency is on the negative real axis in the complex s plane. In the case of a sphere, we note that (1) the current distribution can be represented by real functions (spherical harmonics), and (2) that the current distribution of modes with indices $\ell = 1$, $n = 2$ and $\ell = 2$, $n = 0$ are identical. For spheroids with an arbitrary eccentricity the current distribution is almost real.

From the numerical point of view it is important to know how many natural modes are needed in the sum (4) to maintain a given accuracy. Figure 5 shows the variation with the number of modes of the response at one point on the prolate spheroid. The quantity plotted in this figure is the total axial current $I(z,t)$ defined by $I(z,t) = 2\pi\rho J(\underline{r},t)/bH_o$. The time scale is chosen so that the wavefront hits the scattering object at $t = 0$, the angle of incidence θ is 30°, and $a/b = 0.1$. The accumulated contribution from the first 5 poles in the $\ell = 1$ layer is considered together with a solution labeled "all poles". From Figure 5 and other similar results one can make the following observations

- the fundamental response alone accurately describes the induced current for $t > 10$ b/c,
- the first five modes in the first layer accurately describe the induced current for $t > 4$ b/c ,
- modes in the $\ell \geq 2$ layers only contribute appreciably for $t < b/c$.

3.2 Asymptotic Evaluation of Thin-Wire Response

The response of a thin wire to an incident electromagnetic field can be calculated using the electric-field integral (or integro-differential) equation. The solution of this equation lends itself to an asymptotic expansion in the "antenna parameter" $\Omega = 2 \ln$ (wire length/wire radius). From this solution it can be seen that the natural frequencies of a thin wire are found to be

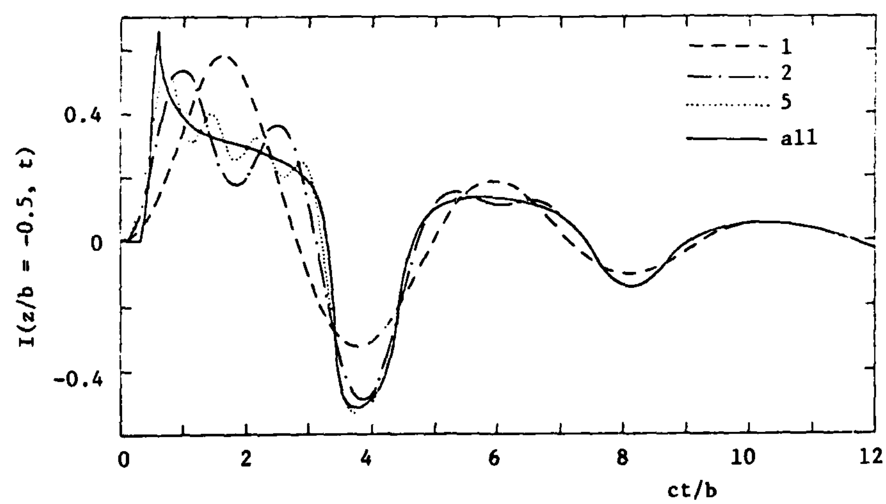

Figure 5. Time response of total current at z/b = -0.5 for 1,2,5 and all mode pairs.

$$s_n = (c/\ell)\{in\pi - \Omega^{-1}[\ell n(2|n|\pi\Gamma) - Ci(2n\pi) + iSi(2n\pi)] + O(\Omega^{-2}), \quad n = \pm 1, \pm 2, \ldots \tag{8}$$

where $\Gamma(= 1.781\ldots)$ is the exponential of Euler's constant, and $Ci(x)$ and $Si(x)$ are the cosine and sine integrals, respectively. Also, to the first approximation, the current distributions of the natural modes are given by

$$I_n(z) = 2\pi a j_n(z) = \sin(n\pi z/\ell) + O(\Omega^{-1}), \quad 0 \le z \le \ell \tag{9}$$

To get some quantitative information about the accuracy of the asymptotic expansion (8) we have in Figure 6 graphed three different representations of the fundamental natural frequency of a straight thin wire, namely

- the asymptotic form (8) correct up to order Ω^{-1} (labeled 1st order approximation),
- an asymptotic form correct up to order Ω^{-2} (labeled 2nd order approximation),
- numerical results.

We note that for $a/\ell = 0.1$ ($\Omega = 9.2$), the natural frequencies calculated from these different methods differ about 20% from each other. We also note that the second-order approximation gives too large a value of the damping constant $|Re\{s_n\}|$ whereas the first-order approximation yields a somewhat too large value of $Im\{s_n\}$.

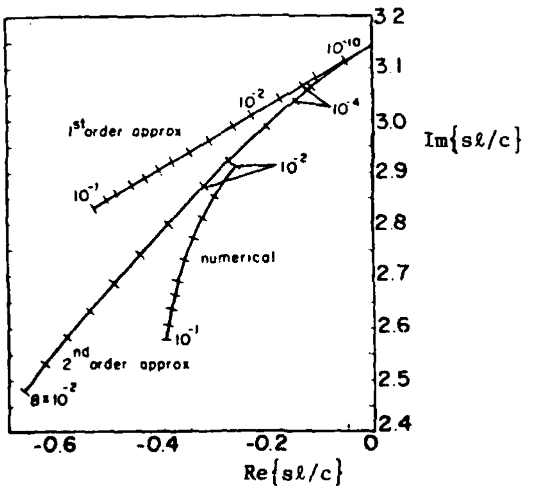

Figure 6. The fundamental natural frequency s_1 for a thin wire. The natural frequencies for $a/\ell = 10^{-10}$, 10^{-5}, 10^{-4}, 10^{-3}, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1 are indicated on the figure.

When the wire is excited by a step-function plane wave whose direction of propagation makes an angle θ with the positive z axis and is so polarized that the electric field vector (strength E_o) makes the angle $\pi/2 - \theta, \theta < \pi/2$, with the positive z axis, one gets the following asymptotic expression for the induced current

$$I(z,t) = (8E_o \ell/\pi\Omega Z_o \sin\theta) U(ct - z\cos\theta) \sum_{n=1}^{\infty} \left\{ \frac{1}{n^2} \right.$$

$$\sin\frac{n\pi z}{\ell}\, [\sin(\omega_n t) - (1)^n \sin(\omega_n t - n\pi\cos\theta)]$$

$$\left. \exp(-\sigma_n t) \right\}, \quad s_n = -\sigma_n + i\omega_n \qquad (10)$$

We note that the time origin is so chosen that the wavefront hits the wire end point $z = 0$ at $t = 0$.

The asymptotic expression (10) was used to numerically calculate on a desk calculator the time history of the induced current at different positions on the wire and at different angles of incidence of the plane wave. A comparison between these results and those obtained from a numerical solution of a space-time domain integral equation is shown in Figure 7 for two angles of incidence, $\theta = 30°$ and $90°$. It is observed in Figure 7 that the asymptotic theory results exhibit faster oscillations than those of the numerical solution. The oscillations are due mainly to the fundamental resonance mode. An inspection of Figure 7 reveals that indeed, the funda-

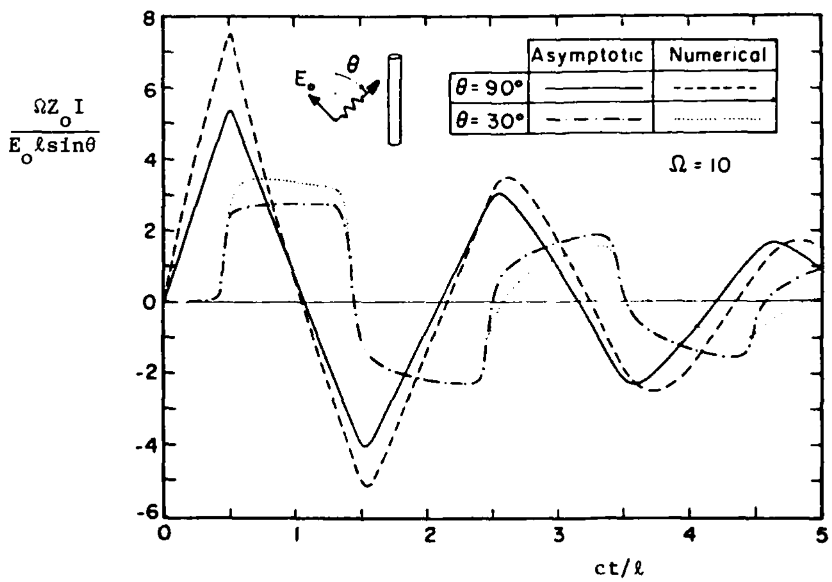

Figure 7. Step function response of the midpoint current for a wire illuminated by a plane wave with electric field strength E_o. The case $\theta = 30°$ and $90°$ are shown. Also included for comparison are the corresponding results obtained by numerically solving a space-time domain integral equation.

mental natural frequency of the asymptotic theory has a larger imaginary part, implying faster oscillations than those obtained by numerically solving the integral equation.

3.3 Aircraft Stick-Model Responses

The relative success of the asymptotic expressions for estimating the response of a simple stick indicates that the same method can be used for more complicated arrangements of intersecting sticks. One example of such an arrangement is the stick model aircraft shown in Figure 8.

Stick models are very useful for estimating the natural frequencies and natural axial current modes of an aircraft. In a stick model, currents of the form

$$I(x) = I_{ind}(x) + A \sinh \gamma x + B \cosh \gamma x \tag{11}$$

are assumed on each of the elements or sticks (Fig. 8), where x denotes a distance coordinate along a given element and A and B are undetermined coefficients. The quantity I_{ind} denotes the current induced on a wire by an incident plane wave whose magnetic vector is perpendicular to the wire and is given by

$$I_{ind}(x) = \frac{-4\pi E_o}{\gamma Z_o \Omega_a \sin\theta} e^{\gamma z \cos} \tag{12}$$

in which $\Omega_a = 2 \ell n$ [(stick length)/(stick radius)], $\underline{\gamma}$ is the propagation vector of the incident field, Z_o is the intrinsic impedance of free space, E_o is the incident electric field strength, and θ is the angle between the propagation vector of the incident wave and the negative unit vector along the stick.

Enforcing appropriate end and junction conditions on the various stick currents leads to a system of linear equations for the unknown current coefficients A,B, etc. The resulting equations may be readily solved to yield the resonance frequencies and natural modes of the simple stick model. The damping constants of the natural modes are found by calculating the radiated power and the time-averaged stored energy of each of the natural modes. For a simple stick, this method results in the same value of the damping constant as the asymptotic method of the previous section does.

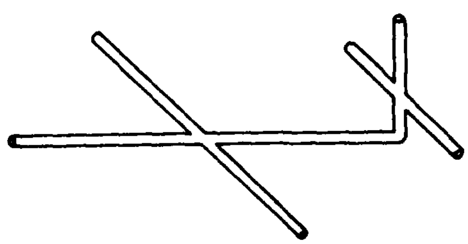

Figure 8. A "simple" stick model.

This type of model has been used to calculate the first several natural frequencies and natural modes for the Boeing 747 and 707 aircraft for symmetric excitation. The natural modes for the 707 model are shown in Figure 9. The natural frequencies for all three aircraft are shown in the complex s-plane in Figure 10.

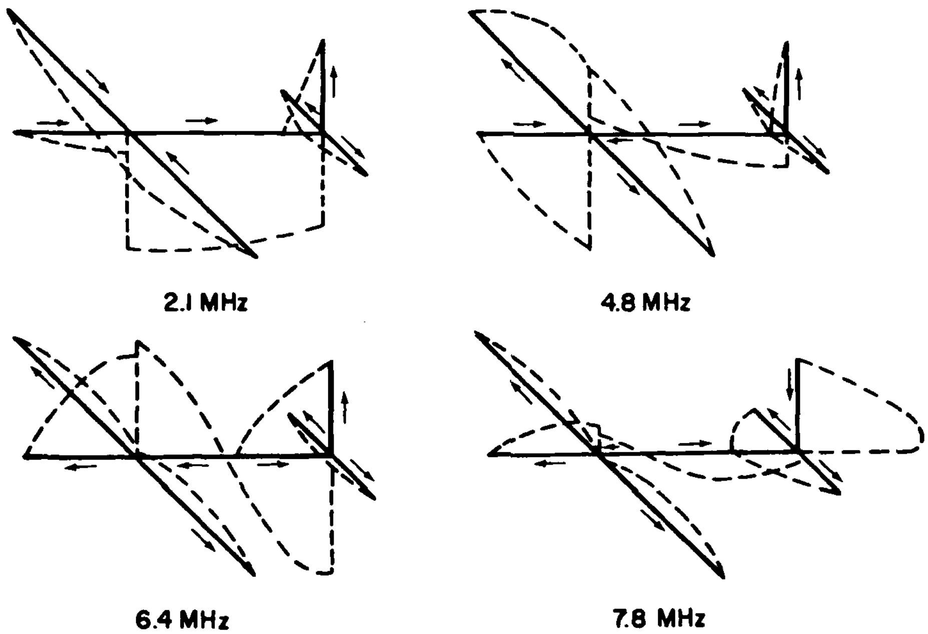

Figure 9. 707 natural modes. The dashed lines represent the current distribution on the aircraft segments at resonance, while the arrows indicate directions of current flow.

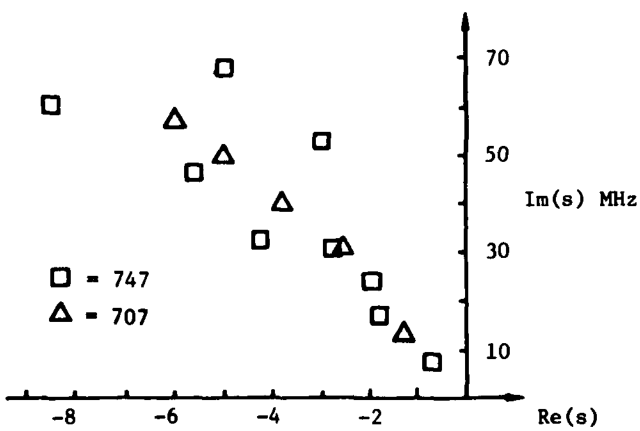

Figure 10. Natural frequencies for 747 and 707 aircraft.

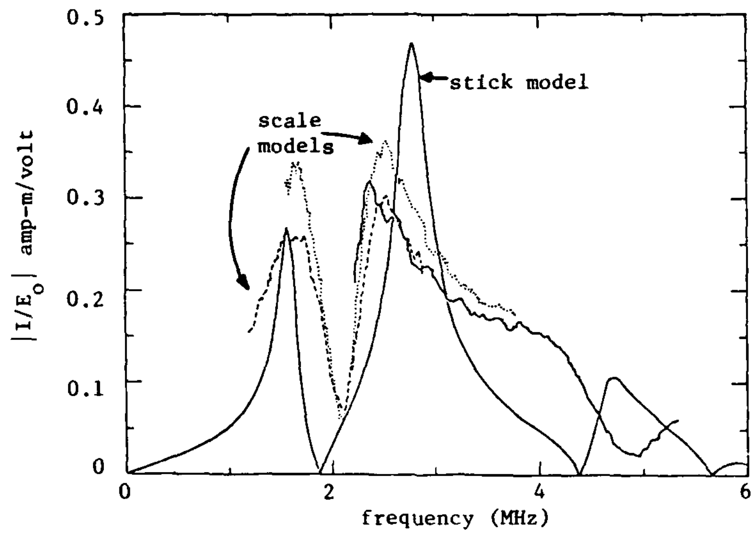

Figure 11. Comparison of the total current on the 747 forward fuselage.

To get some indication about the accuracy that can be obtained from stick model calculations we consider the results shown in Figure 11. In this figure the results from measurements made on two different scale models of the 747 are compared with stick model predictions for the same aircraft. It is observed that

- the resonance frequencies of the two fundamental natural modes are predicted within an accuracy of 10%,
- the resonance peak values are predicted with an accuracy of around 20%.

ON SOME MATHEMATICAL ASPECTS OF SEM, EEM AND SCATTERING

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ABSTRACT

The relationship between the integral equations usually used in SEM and the scattering matrix is examined. Alternate integral equations which exhibit only the poles of the S matrix are given. Examples are used for illustration for a solvable case.

The analytic Fredholm theorem in Banach spaces is discussed and its advantages for numerical calculations emphasized.

The relationship between EEM, SEM and the theory of nonselfadjoint operators is briefly discussed.

INTRODUCTION

The ideas lying behind the Eigenmode Expansion Method (EEM) appear to have been introduced for the first time by Kacenelenbaum in 1969 [6.4]. The Singularity Expansion Method (SEM) was first introduced by Baum in 1971 [3.1] and shortly thereafter independently he introduced EEM [3.4]. The best review paper of these USSR contributions is that due to Voitovic, Kacenelenbaum and Sivov [6.11] and that of the USA's contributions (in this author's opinion) is that of Baum [2.2]. The most complete review of the Russian work through 1976 is the Russian book [6.11] by the above three Russian authors. This book also contains a mathematics appendix by M.S. Agranovic.

A glance at the official bibliography makes it clear that extensive work has been undertaken and completed since these beginnings, and more will be discussed in these preceeding.

In view of the extensive publications the author thought it might be most useful to provide a brief guide to some of the recent mathematical developments without excessive detail and without proofs.

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SEM

For the scalar wave equation an exterior Dirichlet problem would be formally given by the first four of the following equations. The fifth equation is the solution given in terms of generalized eigenfunctions which are distorted plane waves playing the role of the plane waves used in the Fourier integrals which occur when no obstacle is present. The functions $\alpha(k)$ and $\beta(k)$ are related to the initial conditions. This last formula has been rigorously established by Shenk [6.92] in a manner similar to that used by Ikebe [6.38] for the quantum mechanical case. Explicitly the generalized eigenfunctions are defined by (6), (7) and (8). As will be discussed below several different methods are available for the construction of $V_\pm$

(1) $$\Delta U = \frac{\partial^2 u}{\partial t^2}$$

(2) $$U(x,0) = f_1$$

(3) $$\frac{\partial u}{\partial t}(\underline{x},0) = f_2$$

(4) $$U = 0 \quad \text{on} \quad \Gamma$$

(5) $$U(\underline{x},t) = \frac{1}{(2\pi)^{3/2}} \int \phi_\pm(\underline{x},k)\,[\alpha(k)\, e^{ikt} + \beta(k)\, e^{-ikt}]\, d^3k$$

(6) $$(\Delta + k^2)\phi_\pm = 0$$

(7) $$\phi_\pm = 0 \quad \text{on} \quad \Gamma$$

(8) $$\phi_\pm = e^{ik\underline{x}\cdot\underline{\xi}} + V_\pm(\underline{x},k)$$

In contrast to the operator theory approach employing the continuous spectrum SEM employs the Laplace transform which, after a suitable rotation in the s-plane, can be defined by (9), (10), (11), and (12). Condition (10) is one form of the radiation condition which is need to guarantee uniqueness of all k, $\operatorname{Im} k \geq 0$.

(9) $$V(x,k) \triangleq \int_0^\infty U(x,t)\, e^{ikt}\, dt$$

(10) $$\Delta V + k^2 V = -f$$

(11) $$V = 0 \quad \text{on} \quad \Gamma$$

(12) $$\frac{\partial V}{\partial |x|} - ikV = 0(|x|^{-1})$$

The function $V(x,k)$ is sought in terms of the Green's function as given in equation (13). The Green's function in this equation is not the well-known Green's function of free space but is determined by (14), (15), (16), and its domains of analytic and meromorphicity are given by the next two statments (17) and (18). See Dolph, McLeod and Thoe (6.25).

(13) $$V(x,k) = \int_{\Omega} G(x,y,k)f(y)dy$$

where G satisfies

(14) $$(\Delta + k^2)G = -\delta(x-y) \quad \text{in } \Omega$$

(15) $$G = 0 \quad \text{on } \Gamma$$

(16) $$\frac{\partial G}{\partial |x|} - ikG = O(|x|^{-1})$$

(17) $$G(x,y,k) \quad \text{analytic} \quad \text{Im } k \geq 0$$

(18) $$G(x,y,k) \quad \text{meromorphic} \quad \text{Im } k < 0$$

Once $V(y,k)$ has been found the solution of the original problem can be given in terms of the inverse Laplace transform.

If $a > 0.5$, $\text{Im } k > -b$, $b > 0$ and

(19) $$|V| \leq \frac{c}{1 + |k|^a}, \quad |\text{Re } k| \to \infty .$$

Then for $0 < \gamma < b$

$$U(x,t) = \frac{1}{2\pi} \int_{-i\gamma-\infty}^{-i\gamma+\infty} V(x,k)e^{-ikt}\, dk$$

Pushing the contour down yields [6.69]

(20) $$U(x,t) = \sum_{j=1}^{n} e^{-ik_j t} V(x,k_j) + O(e^{-|I_m k_n| t})$$

The function V and k's which occur in this asymptotic formula are the complex eigenfunctions and eigenvalues:

(21) $$\Delta V_j + k_j^2 V_j = 0 \quad \text{in } \Omega$$

(22) $$V_j = 0 \quad \text{on } \Gamma$$

V_j grows exponentially in x .

Several comments are now in order:

(i) The estimate (19) is valid for the Dirichlet problem if the body is (a) star-shaped and (b) non-trapping in the sense of Lax and Phillips (6.52).

(ii) The method is not very useful since it involves the construction of the Green's function and then the determination of its poles.

(iii) It is an open problem to find conditions when the asymptotic series (20) will actually converge.

Instead in SEM it is usual to employ the methods of potential theory. For the exterior time-independent Dirichlet problem corresponding to the time dependent problem we have been considering up until now, this involves consideration of the following set of equations which employ the known Free space Green's function.

$$(\Delta + k^2)V = 0$$

$$V = -V_{inc} \quad \text{on} \quad \Gamma$$

$$\phi_0(x,y) = \frac{1}{2\pi} \frac{e^{ik|\underline{x}-\underline{y}|}}{|\underline{x}-\underline{y}|}$$

$$(23) \quad V(\underline{x},k) = \int_\Gamma \frac{\partial}{\partial n_y} \phi_0(\underline{x},\underline{y})d(\underline{y})d\sigma_y$$

$$(24) \quad I + B(k) \triangleq d(\underline{y}) + \lambda \int_\Gamma \frac{\partial}{\partial n_y} \phi_0(\underline{x},\underline{y})d(\underline{y})d\ _y = 0$$

$$\lambda = \lambda(k) = -1$$

Using the Fredholm alternative the poles are sought as non-trivial solutions of the homogeneous integral equation (24). Those which may occur for real k correspond to eigenvalues of the associated interior Neumann problem. As such, as we shall see, they occur because of the double-layer assumption and can be eliminated by other assumptions. As shown by Dolph and Wilcox, see Dolph [6.96] they do not contribute to the scattered field nor do they appear in it for any separable case.

The homogeneous integral equation which occurs here can be treated mathematically several different ways. Marin [3.9] employed Carleman's Hilbert space theory but the analytic Fredholm theorem attributed to Steinberg [6.94] is perhaps the most convenient since it is applicable in more general Banach spaces. Since matrix approximations are used in the numerical calculation of the poles the choice of the Banach space of continuous functions is perhaps the most convenient. See Dolph and Cho [6.22] for a fuller discussion. For a Hilbert space the proof of the analytic Fredholm theorem can be found in Reed and Simon [6.84].

Analytic Fredholm Theorem - Steinberg [6.94]

Let $O(B)$ = set of bounded operators on the Banach space and let K be an open connected subset of the complex plane. $T(K)$ is analytic in K if for each $k_0 \in K$

$$T(k) = \sum_0^\infty T_n(k-k_0)^n \qquad T_n \in O(B)$$

Theorem. If $T(k)$ is an analytic family of compact operators for $k \in K$, then either $I - T(k)$ is nowhere invertible in Ω, or else $[I-T(k)]^{-1}$ is meromorphic in K.

If B is a separable Hilbert space, the residues are finite rank operators.

One way of eliminating the poles which are not intrinsic to the exterior scattering problem is to replace the Ansatz (23) of the double layer by the complex combination of a double and single layer as used by Brakhage and Werner for the Dirichlet problem [6.11] and by Kussmaul [6.50] for the Neumann problem. In the latter case additional difficulties need to be overcome because of the high order of the singularity.

For the Dirichlet Problem the Ansatz

$$V_+(\underline{x},\underline{k}) = \int_\Gamma \left(\frac{\partial}{\partial n_y} - i\tau\right) \phi_0(\underline{x},\underline{y})\mu(\underline{y})d_y\sigma$$

leads to the homogeneous integral equation:

$$\mu(x) + \int_\Gamma \left(\frac{\partial}{\partial n_y} - i\tau\right) \phi_0(\underline{x},\underline{y})\mu(\underline{y})d_y\sigma = 0 \tag{25}$$

$$\tau(x) = 1 \quad \text{for} \quad \text{Re } k \geq 0$$

$$= 0 \quad \text{for} \quad \text{Re } k < 0 .$$

This equation has only trivial solution for $\text{Im } k \geq 0$ and hence the only non-trivial solutions can occur for $\text{Im } k < 0$ and as Ramm [6.72] has shown these occur at the poles of a Green's function and are in fact the intrinsic poles of SEM.

The non-trivial solutions of this last equation for $\text{Im } k < 0$ also agree with the poles of the S matrix. The S matrix is generally thought to contain all intrinsic properties and in fact Lax and Phillips have given two proofs of the fact that the S matrix uniquely determines the obstacle for the Dirichlet problem -- see Theorem (5.6) cr. [6.52], Chapter V.

For the problem here it can be shown that the $V_-(x,h)$ of (8) and the S matrix are related by the formulas: [In the last equation the integral operator is compact].

$$X = r\underline{\theta}, \ \underline{\xi} = k\underline{w}$$

$$V_-(r\theta,kw) = \frac{e^{-ikr}}{r}\left[s_-(\theta,k,w) + O(1)\right]$$

$$S(k)m(\ ) = m(\theta) + \frac{ik}{2\pi}\int_{|w|=1} m(w)\overline{s_-(\theta,k,w)^*dSw}$$

The complex eigenvalues are poles of $S(k)$.

Derivations of these formulas can be found in Lax and Phillips [6.52], in Schmidt [6.87] for the quantum case of the Schrodinger equation and for a very general case in Shenk and Thoe [6.9]. A physical derivation of the last formula is due to Saxon [6.86] is also contained in Dolph and Cho [6.22]. This last paper also contains an appendix in which a heuristic derivation of the mathematical theory of scattering initiated by Jauch is given.

For the cylinder (24) becomes

$$\mu(a,\phi_0) + \frac{ika}{4}\int_0^{2\pi} \frac{\partial}{\partial(ka)} H_0^{(1)}\left(ka \sin\frac{\theta_0}{2}\right) (a,\mu_0+\phi_0)d\theta_0 = 0$$

and has solutions given by

$$\mu(a,\phi_0) = \sum_{-\infty}^{\infty} \frac{2(-1)^{n-1}J_n(ka)e^{in\phi_0}}{J_n^1(ka)H_n^{(1)}(ka)}$$

The complex roots of the Hankel function are intrinsic, those of the derivative of the Bessel function well-known to be those of the associated interior Neumann problem.

The Brakhage-Werner equation corresponding to (25) is

$$\sigma(a,\phi_0) + \frac{ika}{4}\int_0^{2\pi}\left[\frac{\partial}{\partial(ka)} - i\tau\right] H_0^{(1)}\ \ ka\,\frac{\sin\theta_0}{2}\ \ \sigma(a,\theta_0+\phi_0) = 0$$

with a solution exhibiting only intrinsic poles; namely

$$\sigma(a,\phi_0) = \frac{2}{\pi ka}\sum_{-\infty}^{\infty}\frac{(-1)^{n-1}J_n(ka)e^{in\phi_0}}{H_n^{(1)}(ka)\,[J_n^1(ka) - iJ_n(ka)]} .$$

For this problem the complex eigenfunctions are

$$V(r,\theta) = e^{im\theta}\,H_m^{(1)}(k_0 r)$$

where

$$(\Delta + k_0^2)V = 0, \quad V - 0,\ r = a$$

and the scattering matrix is given, as shown by Shenk and Thoe [6.91] to be

$$S(k)\left(\sum_{-\infty}^{\infty} a_m e^{im\theta}\right) = -\sum_{-\infty}^{\infty}\frac{H_m^{(2)}(ka)a_m e^{im\theta}}{H_m^{(1)}(ka)} .$$

In most cases it is necessary to resort to matrix approximation or to have methods for the calculations of the poles. In the case of the former, Ramm [6.72] has established the following:

> Poles coincide with k_j for which $I + B(k)$ of (24) is not invertible. Let $\{f_j\}$ be an orthonormal basis in $H = L^2(\Gamma)$. Then if
>
> $$\mu_n \triangleq \sum_1^n c_j f_j$$
>
> $$b_{ij} \triangleq \langle [I+B(k)]f_i, f_j\rangle$$
>
> It follows that
>
> $$\sum_{j=1}^n b_{ij}(k)c_j = 0$$
>
> Let $k_m^{(n)}$, $m = 1,2,3\ldots$, be the roots of
>
> $$\det b_{ij}(k) = 0$$
>
> Then the limits $\lim_{n\to\infty} k_m^{(n)} = k_m$ exist and are the poles of the poles of the Greens function G . Every pole of G can be obtain in this way.

EEM

For this same time-independent Dirichlet problem the Eigenmode Expansion method would involve the following:

$$\Delta\phi + k^2 = 0$$

$$\phi = g \quad \text{on} \quad \Gamma$$

$$A(k)\phi = \int_\Gamma \frac{e^{ik|x-y|}}{4\pi|x-y|}\phi(y)dy$$

Ansatz

$$A(k)\phi = g \quad \text{on} \quad \Gamma$$

$$A(k)\phi_n = \lambda_n(k)\phi_n \quad \text{on} \quad \Gamma$$

Picard method gives

$$\phi = \sum_1^\infty \frac{< g,\phi_n >}{\lambda_n}\phi_n$$

when is this valid?

The Picard process is certainly valid for the cylinder and the sphere. In fact as first noted by Kacenelenbaum, Sivov and Voitovic [6.11] for the cylinder they are explicitly given in the case of even θ by

$$\phi_n(\theta) = \cos n\theta$$

$$\lambda_n(k) = \frac{i\pi a}{2} H_n(ka)J_n(ka)$$

$$\phi_n(r, \ ) = H_n^{(1)}(ka)J_n(kr)\cos n\theta \ , \ r \le a$$

$$H_n^{(1)}(kr)\ J_n(ka)\ \cos n\theta \ , \ r \ge a$$

While Dolph [6.20] appears to have been the first to suggest the use of non-self-adjoint operators in scattering problem Agranovic in [6.1], [6.2], [6.3] and Ramm [6.72], [6.73] appear to be the first to systematically apply this idea. Ramm in particular considered the Hilbert space case. That is Let

$$H = L_2(\Gamma)$$

$$< f,g > = \int_\Gamma f(\underline{x})\overline{g(\underline{x})}\ d$$

Then $< Af,g > = < f,\overline{Ag} >$. This is real symmetry and $A \neq A^*$ i.e., A is non-self adjoint.

Question. When is the Ansatz correct? Sufficient condition: $AA^* - A^*A = 0$...(1) i.e., A is normal. Then an orthogonal basis can be found in $H = L_2(\Gamma)$. Here (1) requires

$$\int_\Gamma \frac{\sin k\{|\underline{x}-\underline{t}| - |\underline{t}-\underline{y}|\}}{|\underline{x}-\underline{t}|\ |\underline{t}-\underline{y}|} d\sigma_{\underline{t}} = 0$$

This last condition can be shown to be satisfied by the cylinder and sphere but not for the ellipse or ellipsoid. In particular then any EEM theory results which use the Picard process and are used to construct equivalent circuits are suspect in general.

Before entering into what is known in the case when the operator A is not normal the relation between SEM and EEM when the Picard process is valid should be mentioned.

In [2.2] Baum discussed the matrix case and showed that every zero of $\lambda(k)$ was a pole. More generally:

The Relation between SEM and EEM

Theorem (Ramm). The poles of $G(x,y,k)$ are zeros of the eigenvalues

$$\lambda_n(k) = 0 \qquad G_0 = \frac{1}{4\pi}\,\frac{e^{ik|x-y|}}{|x-y|} \tag{26}$$

$$G(x,y,k) = G_0(x-y) - \int_\Gamma G_0(x,s,k)\mu(s,y,k)\,d\sigma_s \tag{27}$$

where $\mu = \dfrac{\partial G}{\partial n_s}$

$$B\mu = \frac{\partial}{\partial n_x}\,\frac{e^{ik|x-y|}}{2\pi|x-y|}\quad \mu(y)\,d\sigma_y$$

$$\mu + B\mu = 2\,\frac{\partial G_0}{\partial n}$$

If the operator is not normal the situation is much more complicated in general. One usually has to contend with root vectors as they occur in the Jordan normal form. The simplest example of their occurrence is in the matrix solution of ordinary differential equations with repeated roots. The questions of when are the root vectors complete, when do they form a basis are difficult in general. There is one case when there is a simple theorem concerning completeness, namely if the operator is dissipative. An operator A is said to be dissipative if

$$\mathrm{Im} < A\phi,\phi > \;\geq 0 .$$

Many of the operators in mathematical physics are dissipative. For example the free space Green's function is:

$$A\phi = \int \frac{e^{ih|x-y|}}{4\pi|x-y|}\,\phi(y)\,dy .$$

One has for real k

$$\mathrm{Im} < a\phi,\phi > = \iint \frac{\sin k|x-y|}{4\pi|x-y|}\,\phi(x)\overline{\phi(y)}\,d^3x\,d^3y$$

and the delta like behavior of the kernel implies that

$$\mathrm{Im} < A\phi,\phi > \cong \int |\phi(x)|^2 d^3x \geq 0 .$$

A rigourous proof of this can be found in Dolph (6.20).

Ramm (6.72) has established a completeness theorem for such operators which are compact and nuclear.

Before stating his result note that if $S_n(A)$ are the eigenvalues of $(A^*A)^{1/2}$ a compact operator is called nuclear if $\sum_1^\infty S_N(A) < \infty$.

Theorem 1. If $A = P + N$ where P is positive and compact, and N is dissipative and nuclear. Then the root vectors of A are complete.

A simple pertinent example is given by

$$A\phi = \frac{1}{4\pi} \int \frac{e^{ik|x-y|}}{|x-y|} \phi(y) d\sigma_y$$

by taking $P\phi = \frac{1}{4\pi} \int \frac{\phi(y) d\sigma y}{|x-y|}$ and $N\phi = (A-P)\phi$.

More information on root vectors and basic can be found in the reference (6.34).

Finally, space limitations do not permit detailed discussion of many topics important to the further development of this subject. These include the weak perturbation of compact operators see (6.42, (6.55), (6.75) as well as variational principles (6.74) and papers in press by Ramm.

ON THE SINGULARITY AND EIGENMODE EXPANSION METHODS (SEM AND EEM)

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INTRODUCTION

This is a brief summary of the invited talk given by the author at the Lexington (November 1980) meeting. The purpose of this paper is to formulate the mathematical problems important for the SEM and EEM, to answer several basic questions and to draw attention to certain unsolved problems. Some new results are also reported. The detailed presentation of the talk was sent to the Mathematical Notes (ed. C. E. Baum) and submitted for publication in the J. Math. Anal. Appl. The bibliography is not complete: only the papers in which the results mentioned in this article appeared were included in the bibliography.

1. STATEMENT OF THE EEM AND SEM

Let Ω be an exterior domain with a smooth closed boundary Γ, D be the corresponding interior domain, $G_0 = \frac{\exp(ik|x-y|)}{4\pi|x-y|}$, $r = |x|$, $Ag = \int_\Gamma G_0(s,s',k)g(s')ds'$ and $u = \int_\Gamma G_0(x,s,k)g(s)dx$. The function u solves the problem

$$(\nabla^2+k^2)u = 0 \text{ in } \Omega,\ u|_\Gamma = f,\ r(\partial u/\partial r - iku) \to 0 \text{ as } r \to \infty \tag{1}$$

provided that

$$Ag = f \tag{2}$$

If one uses the Laplace transform variable p, then $p = -ik$, and the half plane $\mathrm{Re}\,p > 0$ corresponds to the half-plane $\mathrm{Im}\,k > 0$. Engineers **[6.41] - [2.2]** tried to solve (2) by the formula $g = \sum_{j=1}^{\infty} \lambda_j^{-1} c_j f_j$, where, $Af_j = \lambda_j f_j$, $|\lambda_1| \geq |\lambda_2| \geq \ldots$ and $f = \sum_{j=1}^{\infty} c_j f_j$. This can be done if $A = A^*$ is selfadjoint on $H = L^2(\Gamma)$. The operator A in (2) is nonselfadjoint. Therefore: 1) it may have no eigenvectors (e.g. $Ag = \int_0^x g\,dx$ on $H = L^2[0,1]$),

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2) it may have not only eigenvectors but also root vectors [6.69], [6.68],
3) it is an open question whether one can expand an arbitrary function $f \epsilon H$ in the series of eigenvectors and root vectors of A. Of course one is interested in the rate of convergence of the series in eigen and root vectors and in algorithms for calculation of the root vectors and eigenvalues of A. The outlined method (EEM) has the following merits: 1) instead of problem (1) with a continuous spectrum in the unbounded domain we consider problem (2) with a discrete spectrum on the compact manifold Γ, 2) the resonance properties can be conveniently studied by the EEM. A mathematical study of the EEM was originated in [6.72], [6.73], [6.71].

In order to describe SEM consider the problem

$$u_{tt} = \nabla^2 u \text{ in } \Omega, \; u|_\Gamma = 0, \; u|_{t=0} = 0, \; u_t|_{t=0} = f \qquad (3)$$

The solution of this problem takes the form

$$u = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp(-ikt)v(x,k)dk, \; v = \int_\Omega G(x,y,k)f\,dy \qquad (4)$$

where G is the Green function for problem (1),

$$G = G_0 - \int_\Gamma G_0(x,s,k)\mu ds, \; \mu = \frac{\partial G(s,y,k)}{\partial n_s} \qquad (5)$$

$$[I+T(k)]\mu = 2\frac{\partial G_0}{\partial n_s}, \; T(k)\mu = 2\int_\Gamma \frac{\partial G_0}{\partial n_s} \mu\, ds' \qquad (6)$$

We assume that $f \epsilon C_0^\infty(\Omega)$. From (5), (6) it follows that G is finite-meromorphic in k. This means that G is meromorphic on the whole complex plane k and its Laurent coefficients are degenerate kernels (finite rank operators on H). If $\Omega \subset R^3$ then G is analytic in Im $k \geq 0$. Thus v is meromorphic in k and analytic in Im $k \geq 0$. Let us assume that

$$|v| \leq c(b)(1+|k|)^{-a}, \; a > 1/2, \; |\text{Re } k| \to \infty, \; \text{Im } k = b \qquad (7)$$

where b is an arbitrary const;

$$|\text{Im } k_j| \to \infty \text{ as } j \to \infty, \; |\text{Im } k_1| \leq |\text{Im } k_2| \leq \ldots \qquad (8)$$

where k_j are the poles of v.

Note that (7)$\Longrightarrow$(8). From (7), (8) it follows (by moving the contour of integration in (4) down) that

$$u(x,t) = \sum_{j=1}^N c_j(x,t)\exp(-ik_jt) + o(\exp(-|\text{Im } k_N|t)), \; t \to +\infty \qquad (9)$$

Here $c_j(x,t)\exp(-ik_jt) = \text{Res } v(x,k)\exp(-ik_jt)$ at $k = k_j$, $c_j(x,t) = O(t^{m_j-1})$, where m_j is the order of the pole k_j. Thus we see that (7), (8) and the meromorphic character of v are sufficient for the SEM of the form (9) (asymptotic SEM). It is an open question if the series

$$u(x,t) = \sum_{j=1}^{\infty} c_j(x,t) \exp(-ik_j t) \tag{10}$$

converges. The validity of the EEM was discussed in [6.69], [6.68].

2. COMPLEX POLES OF GREEN'S FUNCTIONS

We saw in Section 1 that complex poles k_j are important. It is interesting to answer the following questions: 1) how does one calculate the poles? 2) are the poles simple? 3) do the poles depend continuously on the scatterer? 4) Can one identify the scatterer from the knowledge of complex poles? 5) what can be said about location of the poles and asymptotic behavior of the large poles nearest to the real axis? 6) are there any monotonicity or other features in the behavior of the purely imaginary poles? 7) What are the properties of the resonant states (natural modes corresponding to the complex poles)? 8) What is the relationship between the poles and the eigenvalues used in the EEM?

We give some answers to the above questions. Three different methods for calculation of the complex poles were given in [6.71-2], [6.68] and [6.74]. The first method is most general. It reduces the problem to calculation of the values k_j at which a certain operator of the type $I + T(k)$, where $T(k)$ is a compact analytic operator function, is not invertible. These k_j can be found by a projection method. The method is described in [6.71-2] (see also [6.68]). Its convergence is proved [6.71] The second method is a variational principle for complex poles: k_j^2 are the stationary values of the functional

$$K(u) = \langle \nabla u, \nabla u \rangle / \langle u,u \rangle, \text{ where } \langle u,v \rangle = \lim_{\varepsilon \to +0} \int \exp(-\varepsilon |x| \ln |x|) uv\, dx \tag{10'}$$

and the integral is taken over Ω. In [6.74] a certain system of test functions was suggested but the rigorous justification of the numerical procedure given in [6.74] is an open problem. In [6.68] a variational principle for the spectrum of compact nonselfadjoint operators was given. In [6.71] it was proved that the complex poles of the Green's functions are the complex zeros of the eigenvalues of certain integral operators. This gives the third method of calculation of the poles: first, one calculates the eigenvalues, then one looks for their zeros. No numerical results are known for the third method. It would be interesting to make numerical experiments and to compare all the three methods.

It is an open question whether the poles are simple. In [6.71] it was proved that the poles are simple if the surface is of such shape that the operator A in (2) is normal, i.e. $AA^* = A^*A$. In [6.73] it was proved that this is so if Γ is a sphere or a straight line (linear antenna). Recently the author gave a simple example of a multiple pole in the problem with third boundary condition:

if $(\nabla^2+k^2)u = 0$ in $r = |x| \geq 1$, $\partial u/\partial r - 2u = \cos\theta$ on $|x| = 1$, $r(\frac{\partial u}{\partial r} - iku) \to 0$ as $r \to \infty$, then $k = -2i$ is a pole of order 2 of $u(x,k)$. Generically multiple poles are exceptions because small perturbations of the shape of the scatterer can destroy multiple poles. On the other hand, since the poles depend continuously on Γ (see [6.68] for precise definitions and proofs) it seems possible that by continuous variation of Γ one can make a multiple pole out of 2 simple poles by merging. Nevertheless, no proof is known that the Green's function of the exterior Dirichlet Laplacian has multiple poles for some Γ.

We have already mentioned that the poles depend continuously on Γ. It is not known whether the set of complex poles determines the scatterer uniquely. A discussion of this question is in [6.76] and [*]. Some information on location of the poles is available: in [6.70] it was proved that the domain $\{\mathrm{Im}\, k < 0,\ |\mathrm{Im}\, k| \leq a\log|\mathrm{Re}\, k| + b,\ a > 0\}$ is free from the complex poles of the Green's function of the Schrödinger operator with a compactly supported potential; in [6.54] a similar result was proved for the poles of the Green's function of the exterior Dirichlet Laplacian; in [6.5] some heuristic arguments are given to show that the domain $\{\mathrm{Im}\, k < 0,\ |\mathrm{Im}\, k| < a|\mathrm{Re}\, k|^{1/3} + b,\ a > 0\}$ is free from the poles of the Green's function of the exterior Dirichlet and Neumann Laplacians provided that Γ is strictly convex and smooth; if Γ is not smooth (say, Γ is a polygon) then there exists a series of poles k_j such that $|\mathrm{Im}\, k_j| = O(\log|\mathrm{Re}\, k_j|)$ as $j \to \infty$ [6.6].

In [6.53] it was proved that there exist infinitely many purely imaginary poles of the Green's functions of the exterior Dirichlet or Neumann Laplacian and

$$cR_1^2 \leq \liminf_{y\to\infty} y^{-2} N(y) \leq \limsup_{y\to\infty} y^{-2} N(y) \leq cR_2^2$$

where $c = 1.138370\ldots\ldots$, $N(y)$ is the number of purely imaginary poles with $|\mathrm{Im}\, k_j| < y$, the obstacle is star-shaped (this means that all points of Γ can be seen from a point in D) and R_1, R_2 are the radii of spheres inscribed in and circumscribing D, respectively. It is pointed out in [*] that if $D_2 = qD_1$, $q > 1$ then $y_j^{(1)} = qy_j^{(2)}$, where $-iy_j^{(1)}$ $(-iy_j^{(2)})$ are the poles of the Green's function of the exterior Dirichlet Laplacian in $\Omega_1(\Omega_2)$, $\Omega_j = R^3\setminus D_j$, $j = 1, 2$, where $R^3\setminus D$ denotes the complement to D in R^3. Therefore in this case $N_2(y) \geq N_1(y)$ and $y_1^{(1)} > y_1^{(2)}$, where $y_1^{(j)}$ are the moduli of the purely imaginary poles with minimal moduli. In [6.53] Theorem 3.5 on p. 751 says that $N_2(y) \leq N_1(y)$. This statement contradicts: 1) the above argument, and 2) the case when D_1 and D_2 are concentric balls and one can calculate $N_1(y)$ and $N_2(y)$ for $y >> 1$ and verify that

$N_2(y) > N_1(y)$. The argument in [6.53] can be used if the assumption $O \subset O_s$ is replaced by the assumption $O \supset O_s$. We mention this because in the literature one can find references and citations of Theorem 3.5 from [6.53] in its wrong form. Using arguments from [6.53] and assuming that D_j, $j=1, 2$ are star-shaped and that $D_1 \subset D_2 \subset D_3$, one can see that $N_1(y) \leq N_2(y) \leq N_3(y)$. Here we used the corrected version of Theorem 3.5 from [6.53]: if $D_1 \subset D_2$ and D_1 is star-shaped, then $N_1(y) \leq N_2(y)$. This theorem is actually proved in [6.53] so that the misstatement of Theorem 3.5 in [6.53] is just a misprint.

Concerning the behavior of the resonant states, that is the solutions of the homogeneous problem (1) for $k = a - iy$, $y > 0$, $f(x) = 0$), satisfying the asymptotic condition

$$u = |x|^{-1} \exp(ik|x|)\sum_{j=0}^{\infty} |x|^{-j} f_j, f_j = f_j(n,y), \ n = x \cdot |x|^{-1}, \quad (11)$$

at infinity, one can prove the following proposition: if $u \exp(-y|x|)|x| \to 0$ as $|x| \to \infty$ then $u \equiv 0$. From this it follows that the resonant states (scattering modes) corresponding to a complex pole $k = a - iy$ grow at infinity exactly as $O(\exp(y|x|)|x|^{-1}$. See also [6.43] Theorem 3. The relationship between SEM and EEM is given in the following proposition ([6.71-2], [6.68]): the set of the complex poles of the Green's function of the exterior Dirichlet Laplacian coincide with the set of complex zeros of the eigenvalues $\lambda_n(k)$ of the operator A defined in (2). It is not known at this time whether the order of a pole can be calculated from the multiplicity of zeros. One can construct other operators with the eigenvalues vanishing at the complex poles [6.68].

3. "ORTHOGONALITY" OF THE EIGENMODES AND RESONANT STATES

By eigenmodes (EM) we mean the eigenfunctions of the operator A defined in (2). This is a nonselfadjoint operator on $H = L^2(\Gamma)$ with the property $[Af,g] = [f,Ag]$, where $[f,g] = (f,\bar{g}) = \int_\Gamma fg\, ds$, $(\cdot,\cdot)$ is the inner product in $L^2(\Gamma)$, the bar denotes complex conjugation. Suppose that $Af_j = \lambda_j f_j$, $[f_j,f_j] \neq 0$, $j = 1, 2, \ldots$ and the set $\{f_j\}$ forms a basis of H. Then any $f \in H$ can be represented as $f = \sum_{j=1}^{\infty} c_j f_j$ and $c_j = [f,f_j]$. This can be proved exactly as in the case of orthogonal Fourier series if one takes into account that $[f_j,f_m] = 0$ for $j \neq m$. The last formula follows from the identity $0 = [Af_j,f_m] - [f_j,Af_m] = (\lambda_j - \lambda_m)[f_j,f_m]$ if $\lambda_j \neq \lambda_m$. If $\lambda_j = \lambda_m$ one can choose f_j, f_m so that $[f_j,f_m] = 0$ for $j \neq m$. Thus the coefficients in the EEM can be easily calculated. If the root vectors are present the formulas for the coefficients in root vectors can also be calculated explicitly [*].

"Orthogonality" of the resonant states corresponding to different complex poles k_1, k_2 holds in the following sense: $<u(x,k_1), u(x,k_2)> = 0$, where $<\cdot,\cdot>$ is defined in (10') (See [6.74] and [*] for details).

4. NONSMOOTH BOUNDARIES

The usual proof of the meromorphic nature of the Green's function of the exterior Laplacian requires smoothness of Γ. Indeed, it is based on the integral equation (6) and on the theorem about the meromorphic nature of the operator $(I+T(k))^{-1}$ [6.80-3]. If this theorem it is assumed that $T(k)$ is a compact operator function analytic in k. If the surface Γ has edges or conical points, the operator $T(k)$ in (6) is no longer compact. Nevertheless the theory is still valid provided that there are no cusps on Γ. This follows from the proposition (see [*] for details): if $T(k) = T + Q(k)$ is an operator function on a Hilbert space H, where $Q(k)$ is analytic in k for $k\varepsilon\Delta$, where Δ is a connected open set in the complex plane, $|T|_{ess} < 1$ and $I + T(k)$ is invertible at some point, then $(I+T(k))^{-1}$ is finite meromorphic in Δ, (finite meromorphic means that the Laurent coefficients are operators of finite rank). By $|T|_{ess}$ we mean $\inf \|T-K\|$, where K runs through the set of all compact operators on H.

It is known [6.12] that $|T(0)|_{ess} < 1$ provided that there are no cusps on Γ. One can now apply the above proposition and conclude that μ in (6) (and therefore G; see (5)) is meromorphic and its Laurent coefficients are degenerate kernels.

5. EXAMPLES, COMMENTS

1. A symmetric (with respect to the form $[f,g]$ defined in section 3) nonselfadjoint operator can have root vectors. Example: $A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$, $[x,y] = x_1y_1 + x_2y_2$. $(A-\lambda I)^{-1}$ has a pole of order 2 at $\lambda = 0$. The corresponding eigenvector is $\binom{1}{i}$ and the root vector is $\binom{1-i}{1}$.

2. The fact that the algebraic problem to which an original integral equation was reduced (e.g. by a projection method) has eigenvalues does not guarantee that the original equation has. (See [6.68], [6.69] and [*] for details and sufficient conditions under which the eigenvalues of the algebraic problem converge to the eigenvalues of the original problem.)

3. The operator $(I+T(k))^{-1}$ can have multiple poles and be diagonalizable (i.e. $T(k)$ has no root vectors).

Example: $T(k) = \begin{pmatrix} -1 + k^2 & 0 \\ 0 & k^2 \end{pmatrix}$, $(I+T(k))^{-1} = \begin{pmatrix} k^{-2} & 0 \\ 0 & (1+k^2)^{-1} \end{pmatrix}$,

$k = 0$ is a pole of order 2.

4. There exists an operator with the root system which forms a basis of H but under a different choice of the root vectors the root system of this operator does not form a basis of H (see [*] for an example).

5. The set of complex poles of the Green's function of the Schrödinger operator does not define uniquely the potential if there are bound states [*].

6. If z is a complex pole of order m of $(I+T(k))^{-1}$, $T(k,\varepsilon)$ is compact and analytic in k and ε for $\{|k-z| \leq a, |\varepsilon| < b\}$ and $T(k,0) = T(k)$ then the poles $z(\varepsilon)$ of $(I+T(k,\varepsilon))^{-1}$ can have a branch point at $\varepsilon = 0$ and ord $z(\varepsilon) \leq m$. Moreover $z(\varepsilon)$ can be represented by Puiseux series, i.e. by a series in powers of $\varepsilon^{1/r}$, where r is some integer (see [*] for details).

7. The multiplicity of the complex poles is not equal to the order of zeros of eigenvalues, generally speaking.

It was proved in [6.72] (see also [6.68]) that the set of complex poles coincide with the set of complex zeros of the eigenvalues of certain integral equations. In the case we are concerned with in this paper one can have in mind the eigenvalues of the equation $[I+T(k)]\, u_j = \lambda_j(k) u_j$, $j = 1, 2, \ldots$. It was an open question whether the orders of the zeros of $\lambda_j(k)$ are equal to the multiplicities of the corresponding poles. We show by presenting an example that this is not so in general. Let us take as $I + T(k)$ a finite dimensional operator with the following matrix

$A(k) = \begin{pmatrix} \lambda(k) & 1 \\ 0 & \lambda(k) \end{pmatrix}$. We have $\lambda_j(k) = \lambda(k)$, $A^{-1}(k) = \begin{pmatrix} \lambda^{-1}(k) & -\lambda^{-2}(k) \\ 0 & \lambda^{-1}(k) \end{pmatrix}$.

If $\lambda(z) = 0$ and m is the order of the zero, then z is the pole of $A^{-1}(k)$ of multiplicity 2m. It is clear from this example that the order of zeros of the eigenvalues will coincide with the multiplicity of the corresponding poles iff $A(k)$ is diagonalizable, that is $A(k)$ has no root vectors. This example is sufficiently general because for a compact T the eigenvalues $\lambda_j \neq -1$ have finite algebraic multiplicites and the corresponding root spaces reduce $I + T(k)$, so that in the root spaces $I + T(k)$ is a matrix operator.

8. Using the ideas given in [6.68] the author proved convergence of the T-matrix approach in scattering theory, widely used in practice.

9. A variational principle for complex poles

In section 3 it was mentioned that the complex poles of the Green function occur at the complex points k at which the homogeneous equation (2) has a nontrivial solution. Let H_q denote the Sobolev space $W_2^q(\Gamma)$, and $|f|_q$ denote the norm in H_q. Consider the variational principle $F(f) = |Af|_1^2 = \min$, $|f|_0 = 1$. If $\{f_j\}$ is a basis of $H = H_0$, and $f^{(n)} = \sum_{j=1}^{n} c_j f_j$, then the problem $F(f^{(n)}) = \min$, $|f^{(n)}|_0 = 1$ yields: $\sum_{m=1}^{n} a_{jm}(k)c_m = 0$, $1 \le j \le n$, $\sum_{j=1}^{n} |c_j|^2 > 0$. Thus (*) $\det a_{jm}(k) = 0$. Let $k_s^{(n)}$ be the complex roots of (*). Then it can be proved that the set of the complex limit points $\{k_s\}$ of the set $\{k_s^{(n)}\}$ coincides with the set of the complex poles of the Green function. This a new result. The functional F(f) is real valued in contrast with the functional K(u) in (10').

Problems

1) Is it true that the root systems of A(k), T(k) form a Riesz basis of H? It is proved that these systems form a Riesz basis with brackets (see **[6.68]** for a proof and definitions). The author thinks that the answer is no.

2) Is there a relation between the order of a complex pole and the multiplicity of the zeros of $\lambda_n(k)$?

3) Can the scatterer be uniquely identified by the set of complex poles of the corresponding Green's function?

4) Prove that there are infinitely many complex poles k_j with $\mathrm{Re}\, k_j \neq 0$ (in diffraction problems and noncentral potential scattering).

5) Are the complex poles of the Green's function of the exterior Dirichlet or Neumann Laplacian simple?

6) Make numerical experiments in the calculation of the complex poles.

7) Prove convergence of the numerical procedure for calculation of the complex poles suggested in **[6.74]**.

8) Find a theoretical approach optimal in some sense to approximate a function f(t) by the functions of the form $f_N = \sum_{j=1}^{N} \sum_{m=1}^{m_j} \exp(-ik_j t)t^{m-1} c_{mj}$. Here the numbers c_{mj}, m_j, k_j are to be found so that f_N will approximate f(t) in some

optimal way. Currently some methods (e.g. Prony method) are used in practice, but they are not optimal. This problem seems to be of general interest (optimal harmonic analysis in complex domain).

9) When can SEM in the form of (10) be justified?

Conclusion

We hope that it was shown in this paper that:

1) EEM is justified (in the generalized form of expansion in root vectors).

2) SEM is justified in the asymptotic form (9).

3) Numerical projection method for calculation of the complex poles is justified.

4) There are many interesting and difficult open problems in the field.

5) Numerical results and experiments are desirable.

Reference

(*) Ramm, A.G., Mathematical foundations of the singularity and eigenmode expansion methods, J. Math. Anal. Appl. (1981).

EVIDENCE THAT BEARS ON THE LEFT HALF PLANE ASYMPTOTIC BEHAVIOR OF THE SEM EXPANSION OF SURFACE CURRENTS

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ABSTRACT

The issues which have persisted in connection with the so-called "entire function contribution" and in connection with alternative coupling coefficient form interrelate closely with the large s asymptotic behavior in the left half plane in SEM representations. To date, no generally applicable rigorous information has been gleaned about this asymptotic behavior. On the other hand, the specific scattering geometries of the sphere and the wire loop yield analytic solutions which can be analyzed asymptotically. Further information can be discerned on a numerical basis or through a procedure based on the discretization of an integral equation. All of this evidence form a mutually-consistent picture of the asymptotic behavior in question. The principal conclusion which results is that the observed behavior taken with the Mittag-Leffler-type expansion theory for complex functions leads to SEM representations which are free from entire function constituents.

1. INTRODUCTION

An issue which has persisted throughout the development of the Singularity Expansion Method (SEM) representation over the last ten years is a question of existence of an entire function constituent in the SEM representation for current on a scattering object. Baum discusses this issue in each of the early papers on SEM (c.f. [2.1][1]) as do Marin and Latham in their rigorous presentation [3.8]. The including of the "possible entire function" in the SEM representation appears to have emerged through an appeal to what we shall term an "interpolative" Mittag-Leffler theorem (c.f. Markushevich[2]). The summand in the representation in the theorem includes polynomials which are introduced by necessity to render the series convergent. Baum chooses to lump formally the sum of all of these polynomials into a single entire function term which he appends to the SEM pole series. To separate these polynomials from the poles to which they correspond jeopardizes, in practice, the convergence of the series.

[1] Bracketed references are given in the SEM bibliography appearing elsewhere in this journal.

[2] A. I. Markushevich, Theory of Functions of a Complex Variable, Vol. 2 trans. R. A. Silverman, Chelsea, New York, 1977, pp. 299-301.

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0272-6343/81/040395-08$2.25

In his development, Baum also introduces an arbitrariness in the SEM representation through the introduction of a "turn-on time" at which the pole series is allowed to begin contributing to the representation of the surface current induced on a scattering object. If this turn-on time is chosen later than the time at which the actual response begins then the entire function contribution to the current representation in the time domain must "fill the gap" between the time the response begins and the time that the pole series contributions in the time domain are allowed to contribute to the representation.

An alternative to choosing a turn-on time in the construction of the SEM representation is to recognize that the surface current response in a scattering problem is unique in the Laplace transform domain for any value of the complex frequency vaiable s which lies on the Bromwich contour. By virtue of its analytic continuability, this representation is unique throughout the complex s plane. The Laplace transform inversion procedure enforces the correct turn-on time of the time domain contributions arising from pole constituents in the transform domain through the large s asymptotic behavior of the current function. This asymptotic behavior is, in general, not the same in the right and left halves of the complex s plane. Through an appeal to Jordan's lemma one either closes a Bromwich contour with a semicircle to the right or to the left obtaining, in the former case, a zero contribution to the current solution or, in the latter case, residue contributions at the poles of the transformed current response. This feature of the Laplace transform inversion procedure is discussed in the SEM context by the author and others[1]. The freedom of choice which one is allowed in the construction of the SEM coupling coefficient as related by Baum in [2.1] appears to be a result of a time during which one is free either to close to the right or close to the left in accord with the asymptotic behavior of the transform current. The particular choice of time at which one switches from a right half plane closure of the Bromwich contour to a left half plane closure dictates a particular form of coupling coefficient.

Mittag-Leffler's work in the expansion of functions in terms of a pole series provides alternative representations depending on the knowledge available about the function to the represented [2,3]. We examine a particular theorem due to Mittag-Leffler which takes advantage of a knowledge of the large s asymptotic behavior of the function to be expanded in the construction of the residue series representation of that function. Under conditions that the function grow asymptotically at an algebraic rate in s, at most, the representation can be cast with only the polynomial constituents in the series as entire function elements. Subsequently, we explore the presently-available evidence about the large s asymptotic behavior or surface current on a scatterer in light of the hypothesis of this "constructive" Mittag-Leffler theorem. In particular, we examine the large s asymptotic behavior of the reciprocal eigenvalues in the expansion of the solutions of electromagnetic scattering from a perfectly conducting sphere and from a thin conducting

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[1] L. W. Pearson, D. R. Wilton and R. Mittra, "Some Implications of the Laplace Transform Inversion on SEM Coupling Coefficients in the Time Domain," to be submitted to Electromagnetics.

[2] G. Mittag-Leffler, "Sur La Representation Analytique des Fonctions Monogenes Uniformes "une Variable Independante," Acta Math. t. 4, 1884, pp. 1-79.

[3] A. R. Forsythe, Theory of Functions of a Complex Variable, Vol. I, Dover, New York, 1965, ch. VII.

circular loop. We also cite a result due to Wilton[1] which, though not rigorous in its development, provides a basis of conjecture that the transform of the surface current density function for a convex perfectly-conducting scatterer is, in general, asymptotically algebraic in the right half of the Laplace transform plane and decays exponentially in the left half plane.

2. MITTAG-LEFFLER THEOREM BASED ON ASYMPTOTIC KNOWLEDGE OF THE FUNCTION

The following theorem is an expanded statement of a theorem due to Mittag-Leffler. Its proof may be found, among other places, in Whittaker and Watson[2]. It is stated here in an expanded form in two senses: first, we expand it to vector valued functions; and second, we include the possibility of algebraic growth with large s on the part of the function. The first extension is, of course, a trivial one and the second one is outlined by Whittaker and Watson. We also observe two corollaries which relate to this theorem.

<u>"Constructive" Mittag-Leffler Theorem</u>

Let $\bar{J}(\bar{r},s)$ be a vector-valued function analytic in s with simple poles $\{s_i\}$, $|s_1| \le |s_2| \cdots < |s_n|$, and with corresponding residues $\{\bar{J}_i(\bar{r})\}$. Let $\{C_m\}$ be a sequence of circles centered at $s=0$ with radii $\{R_m\}$ constructed so the C_m embraces the first m elements of $\{s_i\}$ and such that C_m pases through no poles. If there exists an integer $p \ge 0$ and a (uniform) bound M such that $|s^{-p}\bar{J}| < M$ on C_m as $m \to \infty$, then $\bar{J}(\bar{r},s)$ has the representation

$$\bar{J}(\bar{r},s) = \sum_i \bar{J}_i(\bar{r}) \left[\frac{1}{s-s_i} + \frac{1}{s_i} + \frac{s}{s_i^2} + \cdots + \frac{s^p}{s_i^{p+1}}\right] + \sum_{n=0}^{p} \frac{s^n}{n!} \frac{d^n\bar{J}(\bar{r},0)}{d\, s^n}\,, \tag{1}$$

and this representation is uniformly convergent in s.

<u>Corollary</u> If $\bar{J}(\bar{r},s)$ decays such that $|s\,\bar{J}(\bar{r},s)| < M$ on C_m as $m \to \infty$ then $\bar{J}(\bar{r},s)$ has the representation

$$\bar{J}(\bar{r},s) = \sum_i \bar{J}_i/(s-s_i), \tag{2}$$

which converges uniformly in s.

<u>Corollary</u> If $|s^p\,\bar{J}(\bar{r},s)| \to 0$ on C_m as $m \to \infty$, for $p \ge -1$, then

$$\sum_i \frac{\bar{J}_i(\bar{r})}{s_i^{p+1}} = \begin{cases} \dfrac{d^p\bar{J}(\bar{r},0)}{ds^p}\,, & p \ge 0 \\ 0\,, & p = -1\,. \end{cases} \tag{3}$$

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[1]D. R. Wilton, this issue.

[2]E. T. Whittaker and G. N. Watson, <u>A Course in Modern Analysis</u>, Cambridge, 1927, pp. 134-135.

The effect of the above-stated theorem is, for the class of functions defined by the hypothesis, to obviate the need for an explicit entire function in the SEM representation apart from the convergence polynomial terms in the pole series. Beyond this, the conditions for uniform convergence of the pole series are stated in terms of the polynomials included with each pole term. To construct the series in such a way that it is uniformly convergent renders its termwise inversion to the time domain valid, so that the result is of some practical consequence. The first corollary admits the case of a function which falls off at least as fast as 1/s asymptotically—a case which cannot be handily incorporated into the theorem itself or the associated proof. The second corollary imposes a condition on the summed pole and residue values for terms of order higher than the asymptotic order of the function—a constraint which has proven useful in the case of computations involving the wire loop [4.54].

3. AVAILABLE INDICATIONS OF ASYMPTOTIC BEHAVIOR SCATTERING PROBLEMS

3.1 Introduction

To the author's knowledge, there is no rigorous information available regarding the large s asymptotic behavior of the surface current for a general scattering problem. Since, the high frequency asymptotic limit, localization effects arise, one might be encouraged toward gleaning the needed asymptotic information from physical optics principles. Marin and Latham comment on this in [3.8] and conclude that the physical optics current representation does not apply in the left half of the complex plane. It does lead to the conclusion that the current is asymptotically constant in the right half place.

With asymptotic representations failing, we are forced to turn to specific geometries and to discrete approximate representations of solutions to gain any insight about the applicability of the foregoing theorem. The SEM representation for an electromagnetic scattering problem has been obtained exactly in only two cases to the best of the author's knowledge: the perfectly conducting sphere; and the perfectly conducting wire loop. The former solution is completely rigorous since the sphere geometry is a separable one. The wire loop is analytically tractable with the one approximation that the wire is sufficiently thin that the current may be assumed to be uniformly distributed around the exterior of the wire cross-section.

In each of these cases the solution may be written in terms of a complete uniformly-convergent eigenfunction expansion, provided the source of excitation is located away from the structure in question.[1] Since the solution is written in terms of a uniformly convergent series the series may be integrated termwise when performing the Laplace inversion to the time domain. As a result we pay attention to the large s asymptotic behavior of the reciprocal eigenvalue factors which appear in these terms.

3.2 Scattering from a Sphere

The surface current on a perfectly conducting spherical scatterer of radius a centered at the origin of a coordinate system satisfies an integral equation of the form

[1] Uniform convergence may be demonstrated via the large index behavior of the summand.

$$\int_{-\pi}^{\pi}\int_{0}^{\pi} \bar{\bar{G}}(\theta,\phi|\theta',\phi',s)\cdot\bar{J}(\theta',\phi',s)a^2\sin\theta' d\theta' d\phi' = \bar{\bar{T}}(\theta,\phi)\cdot\bar{E}^{inc}(\theta,\phi,s), \tag{4}$$

where $\bar{\bar{G}}(\theta,\phi|\theta',\phi')$ is the dyadic kernel, $\bar{J}(\theta,\phi,s)$ is the Laplace transform surface current on the sphere $\bar{\bar{T}}$ is a dyadic which selects the tangential component of the Laplace transformed incident electric field $\bar{E}^{inc}(\theta,\phi,s)$. Tai provides an expansion of $\bar{\bar{G}}$ in terms of spherical wave functions. His expansion, when particularized to $r = r' = a$, as above, constitutes an eigenfunction expansion for $\bar{\bar{G}}$ in the eigenfunctions of the integral operator in (4)[1]. From this expansion, we may proceed to resolve (4) as follows.

$$\bar{J}(\theta,\phi,s) = \sum_{n,m}\left\{\frac{1}{\lambda_n^{TM}(s)}\hat{\bar{N}}_{mn}(\theta,\phi)\int_{-\pi}^{\pi}\int_{0}^{\pi}\hat{\bar{N}}_{mn}(\theta',\phi')\cdot\bar{E}^{inc}(\theta',\phi',s)a^2\sin\theta' d\theta' d\phi'\right.$$

$$\left. + \frac{1}{\lambda_n^{TE}(s)}\hat{\bar{M}}_{mn}(\theta,\phi)\int_{-\pi}^{\pi}\int_{0}^{\pi}\hat{\bar{M}}_{mn}(\theta',\phi')\cdot\bar{E}^{inc}(\theta',\phi',s)a^2\sin\theta d\theta' d\phi'\right\}, \tag{5}$$

where $\hat{\bar{M}}$ and $\hat{\bar{N}}$ are normalized eigenfunctions of the integral operator in (4) and where the eigenvalues are given explicitly as

$$\lambda_n^{TM} = \frac{\eta}{2}\left\{\frac{\partial}{\partial r}\left[rj_n(-jsa/c)\right]\frac{\partial}{\partial r}\left[rh_n^{(2)}(-jsa/c)\right]\right\}\Bigg|_{r=a} \tag{6a}$$

and

$$\lambda_n^{TE} = -\frac{\mu s^2a^2}{2c} j_n(-j_v sa/c)h_n^{(2)}(-jsa/c) , \tag{6b}$$

with $c = (\mu\varepsilon)^{\frac{1}{2}}$, $\eta = \sqrt{\mu/\varepsilon}$, and μ and ε the constituitive parameters of the medium.

We may use the large argument asymptotic form for the spherical Bessel function to deduce, with a bit of algebra, the following asymptotic behaviors for the reciprocal eigenvalues appearing in (5).

$$1/\lambda_n^{TM} \sim \frac{4}{\eta}\left[n^2/4+1\right]^{-1}\begin{cases}1 & , \text{ in r.h.p.}\\ e^{2sa/c} & , \text{ in l.h.p.}\end{cases} \tag{7a}$$

while

$$1/\lambda_n^{TE} \sim \frac{4}{\eta}\begin{cases}(-1)^{n+1} & , \text{ in r.h.p.}\\ e^{2sa/c} & , \text{ in l.h.p.}\end{cases} \tag{7b}$$

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[1] C. T. Tai, Dyadic Green's Functions in Electromagnetic Theory, Intext, Scranton, PA, 1971, pp. 168-181.

It is seen that the reciprocal eigenvalues for both the TE and TM constituents of the surface current solution for the perfectly conducting sphere are asymptotically constant for large s in the right half of the complex plane, and they decay exponentially in the left half plane. Therefore they obey the hypothesis of (1) with $p = 0$. The eigenvalues possess a collection of zeros associated with the spherical Hankel function in their respective forms as well as a collection of zeros associated with spherical Bessel function factors. It is well known that these zeros of the Hankel function factors correspond to the exterior (radiating) resonances of the structure and that the (non-radiating) interior resonances manifest the pure imaginary zeros of the Bessel function factors [3.1]. It follows from the reciprocity theorem that only the exterior resonances are excited by a source lying exterior to the sphere.

Application of the "constructive" Mittag-Leffler Theorem of (1) therefore yields a scattering response of the form

$$\bar{J}(\theta,\phi,s) = \sum_{n,m,i} \left\{ R_{ni}^{TM}\left[\frac{1}{s-s_{ni}^{TM}} + \frac{1}{s_{ni}}\right]\hat{\bar{N}}_{mn}(\theta,\phi)\eta_{mni}^{TE} + R_{ni}^{TE}\left[\frac{1}{s-s_{ni}^{TE}} + \frac{1}{s_{ni}}\right]\hat{\bar{M}}_{mn}(\theta,\phi)\eta_{mni}^{TM} \right\}, \tag{8}$$

where $\{R_{ni}^{TE}\}$ and $\{R_{ni}^{TM}\}$ are the residues associated, respectively, with $\{s_{ni}^{TE}\}$ and $\{s_{ni}^{TM}\}$—the complex-valued zeros of $\lambda_n^{TE}(s)$ and $\lambda_n^{TM}(s)$. The η factors are the so-called coupling coefficients

$$\eta_{mni}^{\begin{Bmatrix}TE\\TM\end{Bmatrix}} = \int_{-\pi}^{\pi}\int_{0}^{\pi} \begin{Bmatrix}\hat{\bar{M}}_{mn}(\theta,\phi)\\ \hat{\bar{N}}_{mn}(\theta,\phi)\end{Bmatrix} \cdot \bar{E}^{inc}(\theta,\phi,s_{ni})a^2\sin\theta d\theta d\phi . \tag{9}$$

The derivative of the reciprocal eigenvalue vanishes at $s = 0$.

3.3 Scattering from a Thin Wire Loop

The current on a circular loop of radius b formed from a wire of radius $a \ll b$ may be expressed in an eigenfunction expansion in terms of the eigenfunctions of the thin wire electric field integral equation for the loop[1]

$$I(\phi,s) = -\frac{1}{2\pi}\sum_n \lambda_n^{-1}(s)\, e^{-jn\phi}\int_{-\pi}^{\pi} e^{jn\phi'}\, E_{\phi'}^{inc}(\phi',s)d\phi', \tag{10}$$

where

$$\lambda_n(s) = \eta\, a_n(s)/j2b . \tag{11}$$

The $a_n(s)$ functions are relatively tedious combinations of modified Bessel functions and Lommel Weber functions (c.f. [4.54]. The series (10) is observed to be uniformly convergent based on the large n asymptotic behavior of the series terms as analyzed in King's exposition. Consequently the expansion of the series

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[1] R. W. P. King, "The Loop Antenna for Transmission and Reception," ch. 11 in Antenna Theory, R. E. Collin and F. J. Zucker, eds., McGraw Hill, New York, 1969.

in poles of the λ_n may be viewed on a termwise basis as in the case of the sphere. Umashankar and Wilton have done a careful analysis of the large s asymptotic behavior of the a_n which we require in order to analyze the λ_n. They observe the large s behavior of $a_n(s)$ to be

$$a_n(s) \sim \frac{-jsb}{2\pi c}\left[2\ \ell n(2b/a) - 2\gamma - 2\ \ell n(-jsb/c) - j\pi + (-1)^{n+1}\left(\frac{j\pi c}{s}\right)^{\frac{1}{2}} e^{(2sb/c + j\pi/4)}\right] \tag{14}$$

so that, in particular,

$$a_n(s) \sim \begin{cases} j\ \dfrac{sb}{\pi c}\ \ell n(-jsb/c) & \text{, in r.h.p.} \\ (-1)^{n+1}\ \dfrac{b}{2}\left(\dfrac{s}{\pi c}\right)^{\frac{1}{2}} e^{-2sb/c} & \text{, in l.h.p.} \end{cases} \tag{15}$$

Thus, in (10)

$$1/a_n(s) \sim \begin{cases} \dfrac{-j\pi c}{sb\ \ell n(-jsb/c)} & \text{, in r.h.p.} \\ (-1)^{n+1}\ \dfrac{2}{b}\left(\dfrac{\pi c}{s}\right)^{\frac{1}{2}} e^{2sb/c} & \text{, in l.h.p.} \end{cases} \tag{16}$$

We observe, again, that the reciprocal eigenvalue factors in the current expansion and decay exponentially in the l.h.p. In particular, the *hypotheses* of the corollaries in Section 2 are honored so that

$$I(\phi,s) = -\frac{1}{2\pi}\sum_{n,i}\frac{r_{ni}}{s-s_{ni}}\, e^{-jn\phi}\int_{-\pi}^{\pi} e^{jn\phi'} E^{inc}_{\phi'}(\phi',s)\,d\phi' \tag{17}$$

and

$$\sum_i \frac{r_{ni}}{s_{ni}} = 0 \quad , \tag{18}$$

where

$$r_{ni} = (j\ 2b/\eta)\ \mathrm{Res}\{1/a_n(s)\}\Big|_{s_{ni}} \ .$$

and $\{s_{ni}\}$ are the zeros of $a_n(s)$. The latter observation has been reported by Umashankar and Wilton [4.54], as well.

Before concluding the discussion of the loop, we should comment on a potential weakness in the foregoing argument which is intrinsic to the thin wire approximation leading to (10). Namely, the wire cross-sectional dimension a must be electrically-small—$a \ll c/|s|$. Clearly, as $s \to \infty$, this approximation fails, so that our asymptotic argument is non-rigorous.

3.4 Speculation Relative to a Convex Scattering Object

Wilton has used an approximate approach to the solution of the electric field integral equation in an effort to observe the large s behavior of the solution. He uses a method of moments procedure to cast the integral equation as an approximating matrix equation of the form

$$[\Gamma_{mn}][J_n] = [E_m] \tag{19}$$

and observes the asymptotic behavior of the formal solution

$$[J_n] = [\Gamma_{mn}]^{-1}[E_m] \tag{20}$$

by expanding the inverse in (20) through the use of Kramer's rule[1]. By so doing he is able to isolate explicitly the asymptotic behavior of the inverse matrix for s in the right half and the left half plane. In the course of the development he is forced to impose the condition that the body be convex in order to carry out this asymptotic analysis. Because of the approximate character of this approach it is not clear whether this convexity condition is a hypothesis essential to the discerned asymptotic behavior or whether it is an artifact of the approach. Under the convexity assumption, he concludes that

$$[J_m] \sim [C_m] \begin{cases} P(s) & \text{, in r.h.p.} \\ e^{sL/c} & \text{, in l.h.p.,} \end{cases} \tag{21}$$

where P(s) is a finite polynomial in s and where L is the maximum dimension of the object.

We may conclude from (21) that the expansion (1) applies to $[J_m]$ with some $p \geq 0$. In other words, no explicit entire function except for the convergence polynomials is required in the representation. Certainly, the discretization of the integral equation to the matrix form (19) discards rigor at the point of departure. On the other hand, the accuracy of matrix formulations for engineering purposes is well understood by numerical methods practitioners, and a residue expansion of the solution (20) is quite satisfying over the frequency range where the original formulation (19) is "satisfactorily" accurate. It goes without saying that "satisfactory" often involves subjective judgement. (A similar argument could be used in connection with the thin wire approximation for the loop in the preceding example.)

4. CONCLUSIONS

The application of the Mittag-Leffler Theorem stated in Section 2 is a fruitful basis upon which to base frequency domain forms of the singularity expansion. It is not clear, however, how one might draw generally applicable conclusions for classes of scattering objects. The approach due to Wilton based on the moment method is likely to satisfy some engineering-users of SEM. His approach has not led, to date, to a means of fixing the polynomial order p. That a transcendental entire function, at least, is precluded seems somewhat helpful, however. The observations of Marin and Latham [3.8] based on physical optics indicate that $p = 0$. On the other hand, a high frequency asymptotic approach which yields the necessary information would, indeed, be gratifying.

The specific cases of the sphere and wire loop scatterers work in harmony with the moment method approach to admit the conjecture that, at most, polynomial entire function constituents enter into the singularity expansion in the frequency domain for finite extent perfectly-conducting objects in lossless media. The verity of this conjecture will bear favorably on emerging frequency-domain applications of SEM as well as obviate the concerns with the approximation of entire function constituents in SEM-based equivalent circuits.

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[1] D. R. Wilton, *op. cit.*

LARGE FREQUENCY ASYMPTOTIC PROPERTIES OF RESOLVENT KERNELS

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ABSTRACT

A conjecture on the large complex frequency asymptotic behavior of the resolvent kernel of the electric field integral equation operator is presented. The conjecture is based on a detailed examination of the corresponding large frequency behavior of a matrix approximant to the operator. From this analysis it is concluded that the resolvent decays exponentially on a sequence of concentric circular contours of increasing radius threading between poles in the left half plane. The decay rate is proportional to the distance between observation and source points.

1. INTRODUCTION

In this paper we present a conjecture concerning the asymptotic behavior of the resolvent kernel of the electric field integral equation for large complex frequency s. This asymptotic estimate is needed for deriving correct singularity expansion representations as well as for determining the proper right or left half plane closure times of the Bromwich contour in the Laplace inversion integral.[1] However, attempts to rigorously determine this estimate have not met with success to date. Our conjecture is suggested from an examination of the asymptotic behavior of elements of a matrix approximation to the resolvent and by the fact that both the matrix approximant and the resolvent kernel play similar roles in the solution of a scattering problem formulated as an integral equation. It is presented in the hope that it may stimulate a more rigorous determination of the correct asymptotic behavior.

2. DERIVATION OF LARGE FREQUENCY ASYMPTOTIC BEHAVIOR OF MATRIX APPROXIMANT

Let S denote the surface of a closed, perfectly conducting, convex scatterer. The problem of scattering by S can be formulated in the Laplace transform domain in terms of the integral equation

$$\int_S \tilde{\bar{\bar{\Gamma}}}(\bar{r},\bar{r}',s) \cdot \tilde{\bar{J}}(\bar{r}',s)dS' = \tilde{\bar{E}}^{i}_{tan}(\bar{r},s) \quad , \tag{1}$$

[1] L. W. Pearson, this issue.

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0272-6343/81/040403-09$2.25

where $\tilde{\bar{E}}^i$ is the incident field, $\tilde{\bar{J}}$ is the induced surface current density, the kernel $\tilde{\bar{\bar{\Gamma}}}$ is the free-space Green's dyadic, $\bar{r}$ and $\bar{r}'$ are observation and source points, respectively, on S, and s is the transform variable. The solution of (1) may be formally expressed as

$$\tilde{\bar{J}}(\bar{r},s) = \int_S \tilde{\bar{\bar{\Gamma}}}^{-1}(\bar{r},\bar{r}',s) \cdot \tilde{\bar{E}}^i_{tan}(\bar{r}',s)dS' \tag{2}$$

where $\tilde{\bar{\bar{\Gamma}}}^{-1}$ is the *resolvent kernel*. For a numerical solution, (1) may be approximated by a matrix equation of the form

$$[Z_{mn}(s)][I_n(s)] = [V_m(s)] \tag{3}$$

obtained by the method of moments.[2] The column vector $[I_n]$ contains the coefficients of the expansion through which $\bar{J}$ is approximated via a finite set of basis functions $\bar{f}_n$ as

$$\tilde{\bar{J}}(\bar{r},s) \approx \sum_{n=1}^{N} I_n(s)\bar{f}_n(\bar{r}) = [I_n(s)]^t[\bar{f}_n] \; . \tag{4}$$

The solution of (3) can be expressed in terms of the inverse matrix $[Y_{nm}] = [Z_{mn}]^{-1}$ as

$$[I_n(s)] = [Y_{nm}(s)][V_m(s)] \; , \tag{5}$$

which in turn yields $\tilde{\bar{J}}$ through (4).

For an arbitrarily-shaped surface S, the approach of Rao et al. provides a suitable numerical procedure.[3] In their approach, S is approximated by planar triangular patches, and the basis functions $\bar{f}_n$ are defined on pairs of patches having in common edge n, whose length is ℓ_n. The dipole moment of each basis function $\bar{f}_n$ is $p_n\hat{p}_n$. To establish the connection between the matrix elements $Y_{nm}(s)$ and $\tilde{\bar{\bar{\Gamma}}}^{-1}(\bar{r},\bar{r}',s)$, we suppose that the number of edges N in the triangulation of S is allowed to approach infinity in such a way that the longest edge length approaches zero while each patch normal approaches the local normal of S. We further specify that the centers of edges m and n approach specified points $\bar{r}_m$ and $\bar{r}_n$ while the orientation of the dipole moments of associated basis functions $\bar{f}_m$ and $\bar{f}_n$ approach the direction of specified unit vectors $\hat{p}_m$ and $\hat{p}_n$, respectively, as $N\rightarrow\infty$. Then it is easily shown that

$$\hat{p}_m \cdot \tilde{\bar{\bar{\Gamma}}}(\bar{r}_m,\bar{r}_n,s) \cdot \hat{p}_n = \lim_{N\to\infty} \frac{Z_{mn}(s)}{\ell_m \ell_n p_m p_n} \; . \tag{6}$$

If the numerical procedure converges, i.e., if the right hand side of (4)

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[2] Harrington, R. F., Field Computation by Moment Methods, New York: Macmillan, 1968.

[3] Rao, S. S. M., D. R. Wilton, and A. W. Glisson, "Electromagnetic Scattering by Surfaces of Arbitrary Shape," IEEE Trans. Antennas Propagat., March 1982.

approaches the true surface current $\tilde{\bar{J}}$ as $N\to\infty$, then a similar relationship exists between $\tilde{\bar{\bar{\Gamma}}}^{-1}$ and $[Y_{nm}]$. Our procedure will be to estimate the asymptotic behavior of Y_{nm} for large s as $N\to\infty$ and to assume that the corresponding result holds for $\tilde{\bar{\bar{\Gamma}}}^{-1}$.

Before proceeding, we summarize some particular features of the approximation scheme necessary to the development. The matrix elements Z_{mn} may be written as

$$Z_{mn} = \frac{\eta_o \ell_m}{4\pi} e^{-\frac{s}{c}R_{mn}} \left[\frac{s}{c} \int_{T_n^+ + T_n^-} \bar{f}_n \cdot \left(\frac{\bar{\rho}_m^{c+}}{2R_m^+} + \frac{\bar{\rho}_m^{c-}}{2R_m^-} \right) dS' \right.$$
$$\left. - \frac{c}{s} \int_{T_n^+ + T_n^-} \nabla_S \cdot \bar{f}_n \left(\frac{1}{R_m^+} - \frac{1}{R_m^-} \right) \left(1 + \frac{sR_{mn}}{c} \right) dS' \right]$$
$$= P_{mn}(s)\, e^{-\frac{s}{c}R_{mn}}, \qquad (7)$$

where η_o and c are the intrinsic impedance and the velocity of light in free space, respectively; R_{mn} is the distance between *nodes* m and n, the center points $\bar{r}_m$ and $\bar{r}_n$ of edges m and n, respectively; $T_n^\pm$ are the two triangles common to edge n; $R_m^\pm$ *is the distance between the centroid of* $T_m^\pm$ *and a source* point $\bar{r}'$ in T_n^+ or T_n^-; and $\bar{\rho}_m^{c\pm}$ is the vector from (to) the vertex to (from) the centroid of $T_m^\pm$. Of $\bar{f}_n$, it is sufficient to know only that it is independent of s and hence that $P_{mn}(s)$ is a rational polynomial in s. Eq. (7) differs slightly from the corresponding form of Rao et al.[4] in that the term $\exp(-sR_m^\pm/c)$ has been replaced by the first term of the approximation $\exp(-sR_m^\pm/c) \doteq \exp(-sR_{mn}/c)[1 - s(R_m^\pm - R_{mn})/c]$ in the first integral, while both terms of the approximation are used in the second integral. Use of the approximation is tantamount to neglecting the variation in the propagation factor $\exp(-sR_m^\pm/c)$ observed at the centroid of $T_m^\pm$ for sources in $T_n^\pm$ by replacing it with the propagation factor $\exp(-sR_{mn}/c)$ corresponding to propagation between nodes m and n. In the following we will also rely on the observation from numerical experiments that the convergence of the solution of (5) is independent of the subdivision scheme used to model S, assuming one adheres to the previously mentioned modeling restrictions.

The asymptotic behavior of Y_{nm} is estimated from its definition,

$$Y_{nm}(s) = \frac{(-1)^{m+n}\Delta_{nm}(s)}{\Delta(s)}, \qquad (8)$$

where Δ is the determinant of $[Z_{mn}]$ and Δ_{nm} is the determinant of the matrix obtained by deleting the n^{th} row and m^{th} column of $[Z_{mn}]$. Formally, Δ may be written as

$$\Delta(s) = \sum_{r=0}^{N!-1} (-1)^{\sigma_r} P_{1k_1} P_{2k_2} \cdots P_{N,k_N}\, e^{-sT_r} \qquad (9)$$

where

[4]Rao, S. S. M., D. R. Wilton, and A. W. Glisson, op. cit.

$$cT_r = R_{1k_1} + R_{2k_2} + \cdots + R_{N,k_N} \tag{10}$$

and where each summand with index r corresponds to one of the N! distinct sets $\{k_1,k_2,\ldots,k_N\}$ obtained as permutations of the integers $\{1,2,\ldots,N\}$ and σ_r is the number of inversions in which a larger integer precedes a smaller one in the permutation. Similarly, Δ_{nm} is defined as

$$\Delta_{nm}(s) = \sum_{r'=0}^{(N-1)!-1} (-1)^{\sigma_{r'}} P_{1k_1'}P_{2k_2'}\cdots P_{n-1,k_{n-1}'}P_{n+1,k_{n+1}'}\cdots P_{Nk_N'}\, e^{-sT_{r'}^{nm}} \tag{11}$$

where

$$cT_{r'}^{nm} = R_{1k_1'} + R_{2k_2'} + \cdots + R_{n-1,k_{n-1}'} + R_{n+1,k_{n+1}'} + \cdots + R_{Nk_N'} \tag{12}$$

in which each summand with index r' corresponds to one of the (N-1)! distinct sets $\{k_1',k_2',\ldots,k_{n-1}',k_{n+1}',\ldots,k_N'\}$, with $\sigma_{r'}$ inversions, as permutations of the integer set $\{1,2,\ldots,m-1,m+1,\ldots,N\}$. From (8), (9), and (11), the asymptotic form of $Y_{nm}(s)$ is found to be

$$Y_{nm}(s) = \begin{cases} \mathcal{O}\left[Q_R^{nm}(s)\, e^{-s\left(T_{min}^{nm} - T_{min}\right)}\right], & \mathrm{Re}\, s \to +\infty \\ \mathcal{O}\left[Q_L^{nm}(s)\, e^{-s\left(T_{max}^{nm} - T_{max}\right)}\right], & \mathrm{Re}\, s \to -\infty \end{cases} \tag{13}$$

where Q_R^{nm} and Q_L^{nm} are rational polynomials in s and

$$T_{max} = \max_r T_r \tag{14a}$$

$$T_{min} = \min_r T_r \tag{14b}$$

$$T_{max}^{nm} = \max_{r'} T_{r'}^{nm} \tag{14c}$$

$$T_{min}^{nm} = \min_{r'} T_{r'}^{nm} \;. \tag{14d}$$

From the definition of T_r, Eq. (10), it is seen that determination of $T_{max}(T_{min})$ is equivalent to determining the largest (smallest) value of the N! different sums obtainable by adding together N elements chosen from the array $[R_{ij}]$ such that one and only one element is selected from each row and column. T_{max}^{nm} and T_{min}^{nm} are similarly defined except that the array $[R_{ij}]$ has row n and column m deleted and hence there are only (N-1)! different sums over N-1 elements.

2.1 Determination of T_{max}

To facilitate the determination of T_{max} and T_{max}^{nm}, a special scheme for subdividing S into triangles is used. The scheme requires a different triangulation of S for each element Y_{nm} considered, which, while impractical, is at least possible in principle. It further requires that nodes be placed such that each node $i(\neq m,n)$ can be paired with exactly one other node $j(\neq m,n)$ reached by a *directed line segment* R_{ij} from i to j *passing through the center O of line segment* R_{nm}. The assumption on the convexity of S ensures that such a subdivision scheme is possible. A two-dimensional representation of this scheme is illustrated in Fig. 1.

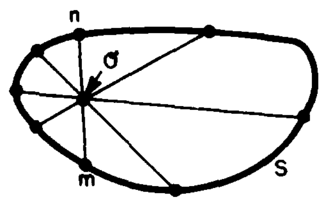

Figure 1. Special subdomain division scheme which pairs nodes by a straight line paths through the center of R_{nm}.

For each r, the elements from the array $[R_{ij}]$ which appear in cT_r may be illustrated pictorially by means of a *line segment diagram*. In the diagram, R_{ij} is shown as a directed line segment *from* node i *to* node j. Thus "from" nodes correspond to rows and "to" nodes correspond to columns of $[R_{ij}]$. In forming T_r for a given r, only one element may be chosen from each row and column of $[R_{ij}]$; hence, in the line segment diagram each node appears as a "from" node once and a "to" node once, as shown in Fig. 2.

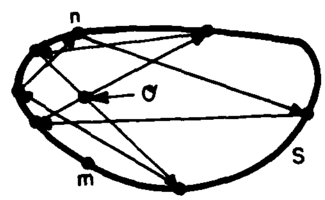

Figure 2. A possible configuration for elements selected from the array $[R_{ij}]$. Note that $R_{mm} = 0$ has been selected.

One possible allowable configuration of line segments is to choose elements R_{nm} and R_{mn} plus those elements R_{ij} and R_{ji} which pass through the center of R_{nm} according to the special subdivision scheme introduced earlier. Indeed, this configuration, which we call the *base configuration*, is found to have a larger sum of segment lengths than any other allowable configuration, and hence the sum is cT_{max}. To see this, we compare the sum of the lengths of the line segments of the base configuration with those of some other allowable configuration, designated as a *modified configuration*, by plotting them both on the same line segment diagram. Only those line segments which differ between the two configurations need be shown since the common segments contribute equally to the sum of lengths. Fig. 3 shows the resultant line segment diagram when the modified configuration is taken to be that of Fig. 2. Note that for every line segment of the modified configuration which leaves, say, node i there must be a base configuration line segment also leaving that node. A similar statement holds for nodes entered by line segments. For example, corresponding to the line segment R_{ij} in Fig. 3 are the segments R_{is} and R_{rj} leaving and entering nodes i and j, respectively. Note also that R_{ij}, R_{iO}, and R_{Oj} form a triangle in which, by the triangle inequality,

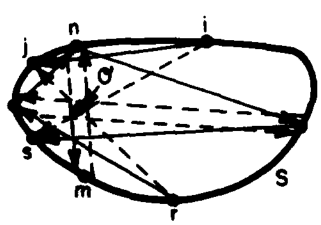

Figure 3. Diagram of line segments which differ between the base configuration (dashed lines) and the modified configuration (solid lines) of Figure 2.

$$R_{iO} + R_{Oj} \geq R_{ij} \quad . \tag{15}$$

Similarly, such triangles are formed by *each* line segment of the modified configuration and the lines from their terminating nodes to *O*. When all the corresponding triangle inequalities are summed, one finds that the sum of the segment lengths in the base configuration exceeds or equals that of the modified configuration. Hence cT_{max} is the sum of the lengths of the segments in the base configuration.

2.2 Determination of cT^{nm}_{max} and asymptotic form of Y_{nm} in the left half plane

The procedure for determining cT^{nm}_{max} is the same as that for determining cT_{max} except that no line segment in either the base or the modified configuration is allowed to leave node n or enter node m since the corresponding row and column, respectively, are missing from $[R_{ij}]$. Hence, by the arguments of the previous section, the base configuration results in the largest sum of segment lengths, cT^{nm}_{max}. This sum, of course, differs from cT_{max} by the length of the deleted segment, R_{nm}, and hence

$$cT^{nm}_{max} - cT_{max} = -R_{nm} \quad . \tag{16}$$

Also, for each segment R_{ij} in the base configuration with $i \neq n$, $j \neq m$, the term $P_{ij}(s)$ appears in the corresponding summand of both (9) and (11) and hence it cancels asymptotically in (8). Thus we have

$$Y_{nm}(s) = \pm \frac{e^{\frac{s}{c}R_{nm}}}{P_{nm}(s)} \quad , \quad \mathrm{Re}\, s \to -\infty \ , \tag{17}$$

where the plus or minus sign is determined by the sign of $(-1)^{m+n+\sigma_{r'}+\sigma_r}$ with r and r' corresponding to the base configurations. An important observation is that the exponent in (17) is independent of the number of segments, N.

2.3 Determination of cT_{min}

Since the diagonal elements of the array $[R_{ij}]$ are all zero ($R_{ii} = 0$), then if diagonal elements only are selected from the array, the sum of the lengths must be zero. Hence we conclude

$$cT_{min} = 0 \quad . \tag{18}$$

In terms of the line segment diagram, every line segment which leaves a node also enters that node and hence is of zero length.

2.4 Determination of cT^{nm}_{min} and asymptotic form of Y_{nm} in the right half plane

The quantity cT^{nm}_{min} is the smallest sum of the line segment lengths that can be formed by selecting one element from each row and column of the array $[R_{ij}]$ with row n and column m deleted. Since deletion of a row and a column generally removes two diagonal elements—leaving N-2 elements on the diagonal, whereas N-1 elements must be selected—then at least one non-diagonal element must be selected. Restricting ourselves initially to the case $m \neq n$, the base configuration is taken to be the one in which all the remaining diagonal elements plus R_{mn} are selected. In Fig. 4 we compare the base configuration with a modified configuration in which a number of non-diagonal elements are chosen, but not R_{mn} nor any element which leaves node n or enters node m. From these restrictions we see that there must be line segment leaving node m and entering some node, say $i \neq m$, from which there must issue still another line segment and so on. This sequence of segments forms a continuous path which may terminate only at node n. The total length of this path is, of course, longer than the direct path R_{mn}. Any other non-zero length line segments of the modified configuration must form a closed path not involving nodes m and n and having a total length which could be reduced to zero by choosing instead the diagonal elements of $[R_{ij}]$ associated with the nodes on the path. Thus the

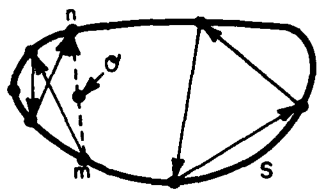

Figure 4. Diagram of line segments which differ from the base configuration (dashed line) for determining cT_{min} and a modified configuration (solid line).

base configuration has the smallest sum of segment lengths,

$$cT^{nm}_{min} = R_{mn} . \tag{19}$$

Returning to the case in which n=m, we note that in this case all the elements of $[R_{ij}]$ may be chosen to be diagonal elements and hence cT^{nm}_{min} is zero. Eq. (19) includes this special case since $R_{nn} = 0$. Hence, from (13), (18), and (19), we conclude that

$$Y_{nm}(s) \longrightarrow \pm \frac{P_{mn}(s)e^{-\frac{s}{c}R_{mn}}}{P_{mm}(s)P_{nn}(s)} , \quad \text{Re } s \to +\infty, \tag{20}$$

where the plus or minus sign is determined by the sign of $(-1)^{m+n+\sigma_{r'}+\sigma_r}$ with r and r' corresponding to the base configurations.

2.5 Asymptotic form of $\tilde{\bar{\bar{\Gamma}}}^{-1}(\bar{r},\bar{r}',s)$

Since $\bar{r}_m$ corresponds to an observation point $\bar{r}$, and $\bar{r}_n$ corresponds to an excitation point $\bar{r}'$, from (17) and (20) we conjecture that the asymptotic behavior of $\tilde{\bar{\bar{\Gamma}}}^{-1}(\bar{r},\bar{r}'s)$ is

$$\tilde{\bar{\bar{\Gamma}}}^{-1}(\bar{r},\bar{r}'s) \longrightarrow \tilde{\bar{\bar{P}}}_{\pm}(s)\, e^{\mp\frac{s}{c}|\bar{r}-\bar{r}'|} , \quad \text{Re } s \to \pm\infty , \tag{21}$$

where the dyad $\tilde{\bar{\bar{P}}}_{\pm}(s)$ is rational in s. As discussed in the following section, the asymptotic behavior of $\tilde{\bar{\bar{\Gamma}}}^{-1}$ in the left half plane must be interpreted as applying on contours which thread between the poles there.

If one inverts $\tilde{\bar{\bar{\Gamma}}}^{-1}(\bar{r},\bar{r}'s)$ so as to obtain its time domain counterpart, $\bar{\bar{\Gamma}}^{-1}(\bar{r},\bar{r},t)$, one concludes from (21) that the Bromwich contour in the Laplace inversion integral may be closed in the left half plane for $t > -|\bar{r}-\bar{r}'|/c$ and in the right half plane for $t < |\bar{r}-\bar{r}'|/c$.[5] Note this implies that in the interval $-|\bar{r}-\bar{r}'|/c < t < |\bar{r}-\bar{r}'|/c$ the contour may be closed in *either* half plane and hence $\bar{\bar{\Gamma}}^{-1}(\bar{r},\bar{r}'t)$ must be zero in this interval.

3. DISCUSSION AND INTERPRETATION

Regardless of whether the asymptotic form of Y_{nm} can be used to infer that of $\tilde{\bar{\bar{\Gamma}}}^{-1}$, as we have assumed here, the derivation of the asymptotic form of the *numerical approximation* to $\tilde{\bar{\bar{\Gamma}}}^{-1}$ of the previous section is rigorous and the result may have some application in the analysis of numerical approximations to the SEM representation. However, the derivation required the assumption of convexity of S and a special scheme for subdividing S which is different for each combination of m and n. It may be useful for further understanding to attempt to remove either or both of these restrictions. (Interestingly, the

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[5] L. W. Pearson, op. cit.

derivations of T_{min} and T_{min}^{nm} do not require *either* restriction.)

The derivation also relies on the finiteness of the discretization. Since the determinant $\Delta(s)$ consists of a finite sum of exponential terms, Re s can always be chosen sufficiently negative that the exponential term with the largest exponent so strongly dominates $\Delta(s)$ that the remaining terms cannot cancel this term to produce a zero. Hence the poles of $Y_{nm}(s)$ are clustered about the Im s axis, whereas in $\tilde{\bar{\bar{\Gamma}}}^{-1}$ the poles are generally distributed throughout the left half of the s-plane. Increasing N in $Y_{nm}(s)$ increases the number of terms in $\Delta(s)$ while decreasing the differences between the exponents of these terms, thereby extending deeper into the left half plane the region where $\Delta(s)$ has zeros. For s lying within this region, however, one might expect that cancellation at zeros changes to constructive addition of terms in $\Delta(s)$ when s lies *between* the zeros. This would imply that exponential growth of $\Delta(s)$ would still occur for points s between the zeros. This is indeed the case for analogous quantities appearing in a number of SEM problems that are analytically tractable, and would imply that the asymptotic estimate remains valid on contours threading between these zeros (i.e., between the poles of $Y_{nm}(s)$). In using the Mittag-Leffler theorem to expand $\tilde{\bar{\bar{\Gamma}}}^{-1}$ or $\tilde{\bar{J}}$, or in determining which half plane to close the Bromwich contour in SEM, these contours are precisely where an asymptotic estimate is required.[6]

The different exponential behavior conjectured for $\tilde{\bar{\bar{\Gamma}}}^{-1}$ in the right and left halves of the s-plane implies a certain arbitrariness in the right and left half plane closure times for the Laplace inverse of expressions involving $\tilde{\bar{\bar{\Gamma}}}^{-1}$. To give a physical interpretation to this phenomenan, we note that the result of inverting $\tilde{\bar{\bar{\Gamma}}}^{-1}(\bar{r},\bar{r}',s)$ is a time-domain Green's function $\bar{\bar{\Gamma}}^{-1}(\bar{r},\bar{r}',t)$ which represents the surface current at a point $\bar{r}$ on S due to a spatial and temporal unit delta-function source applied at point $\bar{r}'$ and at $t = 0$. Physically, this excitation can be approximated by exciting the structure at $t = 0$ with a short pulse produced by a voltage source connected across a small slit in the metallic shell S at $\bar{r}'$. The wavefront produced by the source will expand outward from the source both on the interior and the exterior of the structure. If S is convex, the interior path is the shortest path to the observation point $\bar{r}$ and hence the wavefront will arrive there at $t = |\bar{r}-\bar{r}'|/c$. In the time interval $0 < t < |\bar{r}-\bar{r}'|/c$, the response at $\bar{r}$ is zero, as result which could be obtained either from a valid representation of $\bar{\bar{\Gamma}}^{-1}(\bar{r},\bar{r}',t)$ resulting from the left half plane closure of the Bromwich contour in the inversion integral or from a right half plane closure yielding zero directly. In a more general problem in which the excitation is distributed over S and the response at a point must be calculated from a convolution in both space and time with the excitation, this result may be interpreted as allowing one a choice whether or not to integrate over those excitation points whihc have not had sufficient time to interact with the observation point. This arbitrariness is a cause of the non-uniqueness of coupling coefficients in the SEM-derived time-domain representation.

The conjectured left half plane exponential decay rate implies that the Bromwich contour can also be closed in the left half plane in the time interval $(-|\bar{r}-\bar{r}'|/c,0)$, *before* the source is applied. If this is correct, then the representation of $\bar{\bar{\Gamma}}^{-1}(\bar{r},\bar{r}',t)$ must be zero in this time interval since the response must be causal. In the distributed excitation problem, this result would imply that in the convolution integrals required to compute the response one may actually integrate ahead of the exciting wavefront. Experience with the

[6]Pearson, L. W., this issue.

sphere problem [3.1][7] suggests that the left half plane closure interval may be extended to even earlier times by first performing the spatial convolution with the excitation.

The non-uniqueness of SEM time-domain representations is a factor that is particularly difficult for many electrical engineers to adjust to because of their extensive training in dealing with lumped circuits, where such ambiguities do not arise. The phenomenom is one that is common to distributed parameter systems, however.

4. ACKNOWLEDGEMENT

The author is particularly indebted to L. Wilson Pearson and Raj Mittra for their comments on and discussions of this and related questions concerning the singularity expansion method.

[7] Bracketed references refer to the bibliography at the end of this issue.

SCALAR SINGULARITY EXPANSION METHOD AND LAX-PHILLIPS THEORY

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ABSTRACT

A scalar theory of SEM based on the eigenmode expansion method (EEM) is related to the Lax-Phillips theory of scattering. The Lax-Phillips scattering theory contains results which can be immediately applied to SEM. A byproduct of developing this relationship is a formal proof that SEM poles are simple.

A demonstration of scalar EEM/SEM is presented for scattering by a hard prolate spheroid which includes the sphere as a limiting case. The EEM spheroid solution is shown to have direct bearing on issues of recent concern regarding the validity of certain EEM expansions. The SEM sphere results are shown to contain all of the features of the electromagnetic SEM sphere scattering solution.

1. INTRODUCTION

This paper treats scalar SEM theory as being based on the eigenmode expansions method (EEM) corresponding to the solution of surface integral equations. We prove that the set of complex eigenvalues that play a central role in the Lax-Phillips theory [6.52, 6.53] is exactly the same set as the one consisting of the (nonextraneous) zeros of the eigenvalues of the surface integral equation with the latter set being the SEM pole locations. We also demonstrate that SEM Neumann natural modes are Lax-Phillips eigenmodes evaluated on the surface and that SEM Dirichlet natural modes are normal derivatives of Lax-Phillips eigenmodes. Only the Neumann and Dirichlet problems are treated.

In a previous work [3.12] we focused our attention on scalar SEM corresponding to exterior scattering problems. In this paper we explicitly treat the interior problem as well as the exterior problem and compare interior SEM theory to standard cavity theory as opposed to Lax-Phillips theory. One of the important aspects of Lax-Phillips theory is that it exhibits the great similarity between exterior scattering theory and cavity theory. Throughout the text we refer to the exterior scattering theory and cavity theory as Lax-Phillips theory.

The connection between the Lax-Phillips theory and scalar SEM based on the EEM approach benefits both efforts. The Lax-Phillips approach has a more advanced theoretical foundation. As an example of this, some conditions have been established on the shape of scattering surfaces to which this theory can be applied. In addition, Lax-Phillips theory has been established as a late-time asymptotic theory which includes error estimates. This information had

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not yet been determined by the SEM/EEM approach. In addition, quantitative and qualitative estimating techniques have been developed for the complex eigenvalues. The SEM/EEM approach contributes to the scattering problem by providing explicit expressions for the expansion coefficients in terms of surface quantities. These expansion coefficients, as well as SEM pole locations, have been numerically determined by workers in the EMP community. More generally, workers in the EMP community have developed the capability to numerically obtain SEM solutions for scattering shapes that are beyond analytic treatment.

Finally, we obtained a result made possible by the described connection between scalar SEM and Lax-Phillips theory. We were able to obtain a formal proof that the SEM poles are simple, and this has long been identified as an open question by workers in the EMP community.

To provide a demonstration of scalar SEM for a particular problem, we consider a plane wave incident on a prolate spheroid which then includes the sphere as a special case. We consider the case where Neumann boundary conditions are satisfied on the surface of the spheroid. We then specialize this solution for the case where the spheroid becomes a sphere. We rewrite this scalar sphere solution in a manner which exhibits all of the SEM properties that Baum [3.1] showed for the electromagnetic sphere problem. Having done this, we were immediately in a position to increase our knowledge as a result of treating the scalar problem. The only analytic solution for scattering from a finite object that it is possible to examine in the electromagnetic case is the sphere solution. The eigenmodes for both the electromagnetic and scalar sphere scattering problem do not depend on frequency; however, the scalar spheroid eigenmodes do depend on frequency.

The spheroid EEM solution provided information in another related area. An informative review paper by Ramm [6.69] includes a set of sufficient conditions for the ordinary EEM solution (no root vectors required) to yield a meaningful solution to our scalar integral equation. Ramm presents enough detail in that article for us to conclude that we would not meet the described sufficient conditions unless our scalar integral operator is normal. We are able to show that this is the case when the scatterer is the sphere, and were able to show that this is not the case when the object was the spheroid. For the spheroid, the set of eigenfunctions of the integral operator and its adjoint are clearly not the same sets and are not even simply related through complex conjugation. Despite this, the EEM solution for the spheroid is shown to be the standard separation of variables solution, thus demonstrating that Ramm's conditions are only sufficient but not necessary. This is an important conclusion since much of SEM theory assumes EEM expansions that do not include root vectors.

2. A SCALAR DEMONSTRATION OF EEM/SEM

The presentation of the material in this section is facilitated by referring to Figure 1.

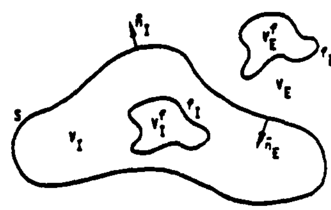

Fig. 1. Separation of space into an interior and exterior region.

In this figure we introduce a surface, S, that separates all of space into an interior region, V_I, and an exterior region, V_E. At this point S is just a mathematically constructed surface; however, as this presentation proceeds, S will correspond to a physical surface on which boundary conditions are satisfied and it will also have shape requirements placed on it. Also, in Figure 1 are sources denoted f_E and f_I, which are nonzero on finite volumes V_E^f and V_I^f contained within V_E and V_I. We are interested in finding solutions to the scalar wave equation in each region.

$$(\nabla^2 - \gamma^2)\tilde{\phi}^{\alpha}_{\beta}(\underline{r},\gamma) = \tilde{f}_{\alpha}(\underline{r},\gamma), \alpha = E,I;\ \beta = N,\ D \tag{1}$$

where the ~ notation indicates Laplace transformation and $\gamma = s/v$ with s being the transform variable and v being the free space speed constant. In the remaining portion of this paper we omit the ~ notation. We are interested in the solution of (1) subject to either Neumann (β = N) ($\partial\phi^{\alpha}_N/\partial n = 0$) or Dirichlet ($\beta$=D) ($\phi^{\alpha}_D = 0$) conditions on S as well as appropriate conditions at infinity. We focus our attention toward obtaining the surface fields with the understanding that the volume fields are readily obtained by performing standard integrals that utilize the surface fields within the integrands.

Standard means yield the following integral equations for the surface fields

$$L^{\alpha}_{\beta}\psi^{\alpha}_{\beta} = h^{\alpha}_{\beta} \quad \beta = N,D\ ;\ \alpha = E,I \tag{2}$$

$$\psi^{\alpha}_N(\underline{r}) = \phi^{\alpha}_N(\underline{r}) \tag{3}$$

$$\psi^{\alpha}_D(\underline{r}) = -\hat{n}_{\alpha}(\underline{r}) \cdot \nabla\phi^{\alpha}_D(\underline{r}) \tag{4}$$

$$L^E_N\phi = \frac{1}{2}\phi(\underline{r}) - \int_S \left(\hat{n}(\underline{r}')\cdot \nabla' G(\underline{r},\underline{r}',\gamma)\right)\phi(\underline{r}')dS' \tag{5a}$$

$$L^I_D\phi = \frac{1}{2}\phi(\underline{r}) - \int_S \left(\hat{n}(\underline{r})\cdot \nabla G(\underline{r},\underline{r}',\gamma)\right)\phi(\underline{r}')\,dS' \tag{5b}$$

$$G(\underline{r},\underline{r}',\gamma) = (4\pi|\underline{r}-\underline{r}'|)^{-1}\exp(-\gamma|\underline{r}-\underline{r}'|)\ . \tag{6}$$

$$\hat{n}(\underline{r}) = \hat{n}_I(\underline{r}) = -\hat{n}_E(\underline{r}) \tag{7}$$

$$L^E_{\beta} + L^I_{\beta} = 1, \quad \beta = N,D \tag{8}$$

The quantities, h^{α}_{β}, are known functions corresponding to the incident fields exited by the sources f_E or f_I in the absence of the scattering surface S.

The formal eigenmode solution for (2) is

$$\psi^{\alpha}_{\beta} = \sum_m \frac{(\psi^{\alpha\dagger}_{m\beta}, h^{\alpha}_{\beta})}{(\psi^{\alpha\dagger}_{m\beta}, \psi^{\alpha}_{m\beta})\ \lambda^{\alpha}_{m\beta}}\psi^{\alpha}_{m\beta}\ , \tag{9}$$

where this solution is valid for those surfaces which require no root vectors as discussed in the Introduction. The condensed summing index, m, allows for degeneracy since we do not preclude the possibility that $\lambda^{\alpha}_{i\beta} = \lambda^{\alpha}_{j\beta}$. The quantities $\psi^{\alpha}_{m\beta}$ and $\lambda^{\alpha}_{m\beta}$ are the eigenfunctions and eigenvalues of L^{α}_{β}. The inner product used in (9) is

$$(f,g) = \int_S f^*(\underline{r})g(\underline{r})dS\ , \tag{10}$$

The quantities $\psi_{m\beta}^{\alpha\dagger}$ are the eigenfunctions of the adjoint operator $L_{\beta}^{\alpha\dagger}$. For the Neumann and Dirichlet operators it can be shown that

$$L_{\beta}^{\alpha\dagger} = L_{\beta'}^{\alpha' *} , \tag{11}$$

where the prime notation on α and β indicates that $\alpha \neq \alpha'$ and $\beta \neq \beta'$ so that the primed indices must take the complementary assignments to the α and β. For example, when $\alpha = E$ and $\beta = N$, (11) states $L_N^{E\dagger} = L_D^{I*}$.

The significance of (9) will be discussed with reference to hard prolate spheroid and sphere scattering. For the prolate spheroid geometry and notation presented in [6.10], we will present the explicit evaluation of the quantities needed to explicitly evaluate the general EEM solution given by (9). The prolate spheroid treated in that reference has the axis of the ellipse oriented along the z axis, and because of the rotational symmetry, an incident plane wave having an arbitrary incident angle, ζ, is given by

$$\phi_{inc} = \exp\left(ik_o(x \sin\zeta + z \cos\zeta)\right) \tag{12}$$

where we have used the relationship $ik_o = -\gamma$ and ϕ_{inc} is a special case of h_N^E. Using the notation $\psi_{mN}^{E} = \phi_{mn{e \atop o}}$, the quantities needed for the EEM solution are

$$\phi_{mn{e \atop o}} = S_{mn}(c,\eta)\begin{pmatrix}\cos m\phi \\ \sin m\phi\end{pmatrix} , \tag{13a}$$

$$\phi^{\dagger}_{mn{e \atop o}} = \left((\xi_1^2-1)/(\xi_1^2-\eta^2)\right)^{1/2} \phi^{*}_{mn{e \atop o}} , \tag{13b}$$

$$(\phi^{\dagger}_{mne},\phi_{inc}) = \pi i^n d^2(\xi_1^2-1)R_{mn}^{(1)}(c,\xi_1)S_{mn}(c, \cos\zeta) , \tag{13c}$$

$$(\phi^{\dagger}_{mno},\phi_{inc}) = 0 , \tag{13d}$$

$$\lambda_{mn} = -ic(\xi_1^2-1)R_{mn}^{(1)}(c,\xi_1)\frac{dR_{mn}^{(3)}(c,\xi_1)}{d\xi_1} , \tag{13e}$$

$$(\phi^{\dagger}_{mne},\phi_{mne}) = \frac{d^2}{4}(\xi_1^2-1)\,\pi N_{mn}(1+\delta_{om}) , \tag{13f}$$

and the notation employed is the same as that used in [6.10], where the meaning can be understood in more detail. Briefly ξ, η, and ϕ are spheroidal coordinates, S_{mn} is an angular spheroidal function, $R_{mn}^{(1)}$ as well as $R_{mn}^{(3)}$ are radial spheroidal functions, d is the distance between the foci of the ellipse, ξ_1 is the constant spheroidal coordinate corresponding to the surface of the spheroid, c is a normalized frequency, N_{mn} is a normalization factor, and δ_{om} is a Kronecker delta function. Obtaining the relations expressed in (13) involves a considerable amount of detailed manipulation and most of this detail is presented in [3.12]. It should be noted that an erratum exists for some of the spheroid material presented in that reference. Substitution of Eqs. (13) into the EEM solution given by (9), readily yields the same solution obtained by separation of variables which is presented in [6.10].

Several important conclusions can be drawn from the spheroid EEM solution. The first is that the EEM solution yields the correct results without the addition of root vectors. This is the case despite the fact that L_N^E for the spheroid is not normal. This follows from the fact the eigenfunctions of the operator and the eigenfunctions of the adjoint operator are clearly different sets. This fact is exhibited in (13a) and (13b). The fact that the spheroid

EEM solution requires no root vectors combined with the lack of normalcy of the associated operator proves that Ramm's conditions [6.69] for no root vectors are not necessary.

Two features of the spheroid solution are worth noting. One feature is that the eigenmodes depend on the frequency as exhibited by the explicit appearance of the "c" factor appearing in (13a). The other feature is that the twofold degeneracy of the eigenvalues (even and odd) is considerably reduced over that of the sphere problem. Finally, a last feature not only relates to the SEM sphere solution, but to the properties of the magnetic field integral equation (MFIE). The operator defined by the MFIE was shown not to be self-adjoint and yet the set of eigenfunctions and adjoint eigenfunctions were shown to be simply related [3.11], in contrast to (13a) and (13b).

The convergence of the EEM solution for the sphere follows from the fact that the spheroid solution converged to the correct answer. The eigenfunctions and adjoint eigenfunctions for the sphere are the same functions which are the products of Legendre polynomials and trigonometric functions

$$\phi_{mn{}^{e}_{o}} = P_n^m(\cos\theta)\begin{matrix}\cos m\phi \\ \sin m\phi\end{matrix} = \phi^{\dagger}_{mn{}^{e}_{o}} \tag{14}$$

and the eigenvalues given in standard Bessel function notation for the sphere having radius a, are

$$\lambda_n = -(\gamma a)^2\, i_n(\gamma a)\, k_n'(\gamma a)\ ,\ j_n(iu) = (i)^n i_n(u)\ ,\ h_n^{(1)}(iu) = -(-i)^n k_n(u) \tag{15}$$

From (14) we can conclude that L_N^E for the sphere is normal. In addition, (14) exhibits the property that the eigenfunctions do not depend on frequency. Equations (14) and (15) together exhibit the degeneracy of the eigenvalues which is a higher order degeneracy than was exhibited by the spheroid. Substituting (14) and (15) into (9) for a plane wave incident field and using orthogonality relations leads to the separation of variables solution for the sphere.

The EEM sphere solution also serves the role of explicitly allowing a demonstration of a scalar SEM solution. First we introduce SEM pole locations. To do this we look for the totality of the zeros of the surface eigenvalues $\lambda_n^{\alpha}(\gamma)$ and we denote an arbitrary zero as $\gamma^{\alpha,T}_{nn'\beta}$ ($s^{\alpha,T}_{nn'\beta}/v \equiv \gamma^{\alpha,T}_{nn'\beta}$) and it satisfies

$$\lambda^{\alpha}_{n\beta}(\gamma^{\alpha,T}_{nn'\beta}) = 0 \qquad n' = 1,2,\ldots,N_T(n) \tag{16}$$

and the superscript T is used to indicate totality. Because of the interrelationship between interior and exterior problems, as exhibited by (11), we expect

$$\left\{\gamma^{\alpha,T}_{nn'\beta}\right\} = \left\{\gamma^{\alpha}_{nn'\beta}\right\} \cup \left\{\gamma^{\alpha,EX}_{nn'\beta}\right\} \quad . \tag{17}$$

The zeros $\gamma^{\alpha}_{nn'\beta}$ are the significant zeros and the zeros having the superscript EX attached are the extraneous zeros. For the exterior problem, the significant zeros are the ones having a negative imaginary part and the extraneous ones are purely imaginary. For the interior problem, the significant ones are purely imaginary and the extraneous ones are in the left half plane. An important result which was derived in [3.12] is

$$\left(\psi^{\alpha\,\dagger}_{n\beta}(\gamma^{\alpha,EX}_{nn'\beta})\ ,\ h^{\alpha}_{\beta}(\gamma^{\alpha,EX}_{nn'\beta})\right) = 0 \tag{18}$$

and this general result is consistent with the SEM sphere solution which is now presented. For the sphere we look for the zeros of the eigenvalue given by (15). The non-extraneous zeros correspond to

$$k_n' \left(\gamma^{E}_{nn'N} a\right) = 0 \tag{19}$$

and the extraneous zeros correspond to

$$i_n \left(\gamma^{E,EX}_{nn'N} a\right) = 0 \tag{20}$$

For completeness we note that λ_n has no zero at $\gamma = 0$. The eigenmode (equivalently separation of variable solution) can be rewritten (no terms added or omitted) for $\zeta = 0$ in (12) as follows

$$\psi^E_N = e^{\gamma a} \sum_{n,n'} \frac{(2n+1)\ (-1)^{n+1}}{a F_n'(\gamma^E_{nn'N} a)} \ \frac{1}{\gamma - \gamma^E_{nn'N}} \ P_n(\cos\theta) \ , \tag{21}$$

where

$$F_n(x) = e^x \, x^2 k_n'(x) \ , \quad F_n'(x) = \frac{dF_n}{dx} \ . \tag{22}$$

We note that the only poles that occur in (21) are the non-extraneous poles and this is consistent with (18). We also note that all of the information that was inferred from the electromagnetic sphere solution presented in [3.1] is contained in (21). A final note is that the concept of natural mode which occurs in SEM appears in (21) in a vacuous manner. This is the case because the SEM natural mode is the eigenmode evaluated at the non-extraneous pole location. As can be seen from viewing (14), no explicit γ dependence occurs for the sphere. In contrast (13a) for the spheroid has the explicit γ dependence for the eigenmode and the SEM significance of the γ dependence has yet to be determined.

3. SEM CONNECTION TO THE LAX-PHILLIPS THEORY

To facilitate the desired connection between the theory just presented and the Lax-Phillips theory which is a volume approach, we introduce the volume eigenvalue equations for both the interior and exterior scalar scattering problems

$$(\nabla^2 - \gamma^{\beta 2}_{\alpha n}) \phi^{\beta}_{\alpha n} = 0 \qquad \alpha = E, I \qquad \beta = N, D \ . \tag{23}$$

For the interior problem, either Neumann or Dirichlet boundary conditions on the surface as well as certain volume behavior requirements, e.g., requirements which force us to reject the explicit solutions obtainable for separable coordinates that become unbounded, are known to lead to denumerable sets of eigenfunctions $\{\phi^{\beta}_{In}\}$ and eigenvalues $\{\gamma^{\beta}_{In}\}$.

Returning to Eq. (23), we consider the exterior scattering problem. The fact that, subject to appropriate boundary conditions, there exists only a denumerable set of eigenfunctions and eigenvalues is not as well known to be the case for the exterior problem as it is for the interior problem. Either the Neumann or Dirichlet boundary conditions together with the γ-outgoing condition expressed as

$$\phi^{\beta}_{En} \sim K^{\beta}_n(\theta, \phi) r^{-1} e^{-\gamma^{\beta}_{En} r} \tag{24}$$

for large r lead to the denumerable sets. As discussed in [3.12], we also have the strict inequality

$$\mathrm{Re}\,\gamma^{\beta}_{En} < 0 \quad . \tag{25}$$

The requirement on the shape of the scatterer is an issue that has received attention. Many of the cited properties have been proved when the object is star-shaped, i.e., a point within the object can be found from which a straight line can be drawn that connects this point to any other point within the volume bounded by the surface of scatterer. Star-shaped surfaces include convex surfaces. There has been some work and some conjecture concerning the applicability of the work to surfaces described as confining or nonconfining [6.53]. It should be noted that the predominant situation of an imperfectly sealed enclosure, i.e., a finitely thick-walled enclosure containing an aperture, is not a star-shaped surface.

We now restrict our attention to surfaces for which we have the desired discrete spectrum for the exterior scattering situation and utilize the identity

$$\nabla\cdot[\phi^{\beta}_{\alpha n}\nabla G - G\nabla\phi^{\beta}_{\alpha n}] = \phi^{\beta}_{\alpha n}\nabla^2 G - G\nabla^2\phi^{\beta}_{\alpha n} \quad . \tag{26}$$

Substituting (23) into (26) and utilizing $(\nabla^2 - \gamma^2)G = -\delta(\underline{r}-\underline{r}')$ we have

$$\nabla\cdot[\phi^{\beta}_{\alpha n}\nabla G - G\nabla\phi^{\beta}_{\alpha n}] = (\gamma^2 - \gamma^{\beta\,2}_{\alpha n})\phi^{\beta}_{\alpha n}G - \delta(\underline{r}-\underline{r}')\phi^{\beta}_{\alpha n} \quad . \tag{27}$$

First we consider this equation for the interior Neumann problem and integrate both sides over the interior volume. We next use the divergence theorem as well as the Neumann boundary condition to obtain

$$\phi^{N}_{In}(\underline{r}') + \int_S \hat{n}_I(\underline{r}) \cdot \nabla G(\underline{r},\underline{r}')\,\phi^{N}_{In}(\underline{r})dS = (\gamma^2 - \gamma^2_{In})\int_{V_I}\phi^{N}_{In}\,G dV \quad . \tag{28}$$

Interchanging the notation $\underline{r}$ and $\underline{r}'$ and taking the limit as $\underline{r}$ approaches the surface, we have

$$L^{I}_{N}\phi^{N}_{In} = (\gamma^{N^2}_{In} - \gamma^2)\int_{V_I}\phi^{N}_{In}\,G dV' \quad . \tag{29}$$

where L^{I}_{N} is defined in section 2. A similar treatment for the interior Dirichlet problem can be readily performed. We summarize the results of (29) and the results of the similar Dirichlet treatment as follows

$$L^{I}_{\beta}y^{\beta}_{In}(\underline{r}) = (\gamma^2 - \gamma^{\beta 2}_{In})\,F^{\beta}_{In}(\gamma, \gamma^{\beta}_{In}, \underline{r}) \tag{30}$$

where

$$y^{N}_{\alpha n} = \phi^{N}_{\alpha n} \tag{31}$$

and

$$y^{D}_{\alpha n} = -\hat{n}_{\alpha}\cdot\nabla\phi^{D}_{\alpha n} \tag{32}$$

Equation (29) defines F^{β}_{In} for $\beta = N$ and a somewhat similar expression for F^{D}_{In} is readily obtainable; however, for our purposes we do not need the explicit expression. We only require the fact that F^{β}_{In} is finite and non-zero at $\gamma = \pm\gamma^{\beta}_{In}$. Equation (30) plays an important role in this paper and the comparable equation for the exterior region plays an even more important role. The derivation of the comparable exterior equation is far more intricate than the derivation of (30). It utilizes an involved bounding argument that requires the use of the outgoing condition (24), special attention to the behavior of the Green's function, and several intricate manipulations. These details can be found in [3.12]. The resulting equation is

$$L^E_\beta\, y^\beta_{En}(\underline{r}) = (\gamma - \gamma^\beta_{En})\, F^\beta_{En}(\gamma, \gamma^\beta_{En}, \underline{r}) \tag{33}$$

and F^β_{En} is bounded and non-zero for $\gamma = \gamma^\beta_{En}$ and is non-zero for $\gamma = -\gamma^\beta_{En}$.

We are now in a position to draw the comparison between scalar SEM and Lax-Phillips theory. These conclusions will be drawn in terms of sets which are now defined. One set is $\{\gamma^\alpha_{nn',\beta}\}$ which is defined by (16) and (17) together with the defining relationship $\gamma^\alpha_{nn',\beta} = s^\alpha_{nn',\beta}/v$. The other set is the one consisting of the Lax-Phillips eigenvalues which are denoted $\{\gamma^\beta_{\alpha n}\}$. We will prove

$$\left\{\gamma^\alpha_{nn'\beta}\right\} = \left\{\gamma^\beta_{\alpha n}\right\} \quad . \tag{34}$$

The proof that

$$\left\{\gamma^\beta_{\alpha n}\right\} \subset \left\{\gamma^\alpha_{nn'\beta}\right\} \tag{35}$$

follows from the bounding arguments just presented.

Substituting $\gamma = \gamma^\beta_{En}$ into (34) leads to the eigenvalue equation that implies (35) for the exterior problems. Substituting $\gamma = \pm\gamma^\beta_{In}$ into (30) leads to the eigenvalue equation that implies (35) for the interior problems. It is significant that the exterior bounding argument did not allow us to draw corresponding conclusions for $\gamma = -\gamma^\beta_{En}$. This is in agreement with the fact that exterior Lax-Phillips eigenvalues are strictly in the left-half plane and we would have inconsistent results if we could imply the existence of right-half plane values for $\gamma^E_{nn'\beta}$. The fact that for the interior problem, we could conclude the existence of values of $\gamma^I_{nn'\beta}$ corresponding to $-\gamma^\beta_{In}$ is to be expected. This is the case because the γ^β_{In}'s correspond to cavity resonances and fall on the imaginary axis. The existence of plus and minus γ^β_{In} being in the set simply implies that complex conjugate pairs occur in the set $\{\gamma^I_{nn'\beta}\}$.

In order to prove that

$$\left\{\gamma^\alpha_{nn'\beta}\right\} \subset \left\{\gamma^\beta_{\alpha n}\right\} \tag{36}$$

we will construct appropriate functions involving surface integrals and will show that these functions are Lax-Phillips eigenfunctions. The Lax-Phillips eigenvalues associate with these constructed eigenfunctions will be seen to be the $\gamma^\alpha_{nn'\beta}$'s. We now form

$$X^N_{n\alpha}(\underline{r}_v) = \int_S G(\underline{r}_v, \underline{r}', \gamma^\alpha_{nn'N})\, \psi^{N+*}_{n\alpha}(\gamma^\alpha_{nn'N}, \underline{r}')\, dS' \tag{37}$$

where $\psi^{N+*}_{n\alpha}(\gamma^\alpha_{nn'}, \underline{r})$ is the complex conjugate of the adjoint eigenfunction appearing in (9), but evaluated at a pole location, and G is the free space Green's function given in (6) with $\underline{r}$ replaced by $\underline{r}_v$. Direct substitution will show that

$$(\nabla^2_V - \gamma^{\alpha 2}_{nn'N})\, X^N_{n\alpha} = 0 \ . \tag{38}$$

In addition $X^N_{n\alpha}$ can readily be seen to satisfy the γ-outgoing condition and X^N_{nI} is finite in V_I. It remains to show that the Neumann boundary conditions on S are satisfied. To show this we take the gradient of both sides of (37) to obtain

$$\nabla_v \chi^N_{n\alpha}(\underline{r}_v) = \int_S \nabla_v G(\underline{r}_v, \underline{r}', \gamma^{\alpha}_{nn'N}) \ \psi^{\alpha\dagger *}_{nN} \ (\gamma^{\alpha}_{nn'N}, \underline{r}') \, dS' \tag{39}$$

Identifying $\underline{r}$ as a point on S which will be approached by $\underline{r}_v$, and defining $\hat{n}_\alpha(\hat{\underline{r}})$ as the appropriate normal defined at the point $\underline{r}$, consistent with the convention depicted in Figure 1, we have

$$\hat{n}_\alpha(\underline{r}) \cdot \nabla \chi^N_{n\alpha}(\underline{r}) = L^{\alpha\ \dagger *}_N(\gamma^{\alpha}_{nn'N}) \, \psi^{\alpha\ \dagger *}_{nN} = \lambda^{\alpha}_{nN}(\gamma^{\alpha}_{nn'N}) \psi^{\alpha\dagger *}_{nN} \tag{40}$$

Referring to the hypothesis that $\gamma^{\alpha}_{nn'N}$ is a zero of λ^{α}_{nN}, we conclude that

$$\frac{\partial \chi^N_{n\alpha}(\underline{r})}{\partial n} = 0 \ . \tag{41}$$

Equations (38) and (41) prove (36) for the Neumann problem. For the Dirichlet problem we construct

$$\chi^D_{n\alpha}(\underline{r}_v) = \int \hat{n}_\alpha(\underline{r}') \cdot \nabla' G(\underline{r}_v, \underline{r}', \gamma^{\alpha}_{nn'D}) \, \psi^{\alpha\dagger *}_{nD}(\gamma^{\alpha}_{nn'D}, \underline{r}') \, dS' \tag{42}$$

Direct substitution shows that

$$(\nabla^2_v - \gamma^{\alpha\,2}_{nn'D}) \chi^D_{n\alpha} = 0 \tag{43}$$

and χ^D_{nE} satisfies the γ-outgoing condition while χ^D_{nI} is finite in V_I. Taking the limit as $\underline{r}_v$ approaches the surface and using the adjointness relationship given by (11) we obtain

$$-\chi^D_{n\alpha} = (L^{\alpha\dagger}_D \psi^{\alpha\ \dagger}_{nD})^* = \lambda^{\alpha}_{nD}(\gamma^{\alpha}_{nn'D}) \psi^{\alpha\dagger *}_{nD} \ . \tag{44}$$

The definition of $\gamma^{\alpha}_{nn'D}$ together with (44) shows that $\chi^D_{n\alpha}$ satisfies the Dirichlet condition and this fact together with (43) proves (36) for the Dirichlet problem.

From the construction that led to the proof of (34) it follows that the Lax-Phillips eigenfunctions evaluated on the surface are related to SEM natural modes, as follows

$$\left\{ y^{\beta}_{\alpha n}(\underline{r}) \right\} = \left\{ \psi^{\alpha}_{n\beta}(\gamma^{\alpha}_{nn'\beta}, \underline{r}) \right\} \tag{45}$$

4. FORMAL PROOF OF SIMPLE POLES

The important structure of equations already presented in this paper is as follows.

$$L^{\alpha}_{\beta}(\gamma) \ y^{\beta}_{\alpha n}(\underline{r}) = f^{\alpha}_{n\beta}(\gamma, \underline{r}) \tag{46}$$

and this equation summarizes (30) and (33). The significant aspect of $y^{\beta}_{\alpha n}(\underline{r})$ is that it does not depend on γ. The quantities $f^{\alpha}_{n\beta}(\gamma, \underline{r})$ represent a condensed notation for the right-hand sides of the cited equations. Recalling the previous discussions of these right-hand sides, $f^{I}_{n\beta}(\gamma, \underline{r})$ has simple zeros at $\gamma = \pm\gamma^{\beta}_{In}$ and these occur on the imaginary axis so these zeros represent a complex conjugate pair. For the exterior problem, $f^{E}_{n\beta}(\gamma, \underline{r})$ has a simple zero at $\gamma = \gamma^{\beta}_{En}$ but not at $\gamma = -\gamma^{\beta}_{En}$ and γ^{β}_{En} has a negative real part. We now formally employ the eigenmode expansion given by (9) to obtain

$$y^{\beta}_{\alpha n}(\underline{r}) = \sum_m \frac{\left(\psi^{\alpha\dagger}_{m\beta}(\gamma), f^{\alpha}_{n\beta}(\gamma)\right)}{\left(\psi^{\alpha\dagger}_{m\beta}(\gamma), \psi^{\alpha}_{m\beta}(\gamma)\right)\lambda^{\alpha}_{m\beta}(\gamma)} \psi^{\alpha}_{m\beta}(\gamma, \underline{r}) \tag{47}$$

Next, we substitute $\gamma = \pm\gamma^{\beta}_{In}, \gamma^{\beta}_{En}$ into this equation and note that the left-hand side of this equation is finite and nonzero while $f^{\alpha}_{n\beta}(\gamma)$ has a simple zero corresponding to these values of γ. We avoid a contradiction by first noting that $\lambda^{I}_{m\beta}(\pm\gamma^{\beta}_{In})$ and $\lambda^{E}_{m\beta}(\gamma^{\beta}_{En})$ are zero for some respective m's, and this follows from (34). If the order of these zeros were less than one, then the right-hand side would still be zero for these values of γ; and if the order were greater than one, then the right-hand side would be infinite at these values of γ. Since the left-hand side is finite and nonzero, we conclude that the zeros of the eigenvalues must be simple. As discussed in Section 2, the zeros of the eigenvalues are the SEM poles and we have just presented a formal proof that they are simple.

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EFFECT OF CHANGES IN FUNDAMENTAL SOLUTIONS ON SINGULARITIES OF THE RESOLVENT

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Time harmonic scattering of acoustic and electromagnetic waves from impenetrable obstacles leads to boundary value problems which are uniquely solvable for values of wave number, k, with non negative imaginary part. However existence and uniqueness questions remain when the imaginary part of k is negative. In fact the SEM poles comprise such a set of values of k. Integral equation formulations of scattering problems involving smooth closed scatterers introduce additional exceptional values of k, those corresponding to eigenvalues of an adjoint interior problem. These additional characteristic values of k are real and pose a serious obstacle to numerical solutions of integral equations for exterior problems. In recent years, considerable attention has been directed to resolving the problems present at interior eigenvalues of which only a representative sample is cited [6.8, 6.11, 6.13, 6.47, 6.49, 6.56, 6.63, 6.106, 6.122].

A method which involves modifying the free space Green's function in the derivation of a boundary integral equation has been proposed by Jones [6.40]. Ursell [6.107] clarified parts of Jones' work and Kleinman and Roach [6.46] extended the results to three dimensional scalar problems, and provided explicit choices of the modification which optimized the formulation with respect to various criteria (e.g. minimizing the spectral radius of the integral operator). While the Jones modififcation eliminates real exceptional values of k by shifting them into the complex plane, a question remains as to the effect of the modification on the SEM poles. This note examines this question in the case of scalar three dimensional scattering with both Dirichlet and Neumann boundary conditions. First some results on boundary integral equation formulations with modified Green's functions are cited. Then the case of scattering from a sphere is examined in detail and we conclude with some comments on more general scatterers.

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1. BOUNDARY INTEGRAL EQUATIONS

We adopt the notation of [6.46] and let D_- denote a connected bounded domain in $\mathbb{R}^3$ with smooth boundary ∂D and exterior D_+. Erect a Cartesian coordinate system with origin in D_-, and let p and q denote points in $\mathbb{R}^3$ and $R(p,q)$ be the distance between them. Furthermore define normalized spherical wave functions

$$(1)\quad c^e_{nm}(p) := \left\{\frac{-ik}{2\pi}\epsilon_m(2n+1)\frac{(n-m)!}{(n+m)!}\right\}^{1/2} h_n^{(1)}(kr_p)P_n^m(\cos\theta_p)\cos m\phi_p$$

$$(2)\quad s^e_{nm}(p) := \left\{\frac{-ik}{2\pi}\epsilon_m(2n+1)\frac{(n-m)!}{(n+m)!}\right\}^{1/2} h_n^{(1)}(kr_p)P_n^m(\cos\theta_p)\sin m\phi_p$$

where (r_p,θ_p,ϕ_p) are the spherical polar coordinates of p. Now we denote the fundamental solution of the Helmholtz equation or unmodified free space Green's function as

$$(3)\quad \gamma_o(p,q) := -\frac{e^{ikR(p,q)}}{2\pi R(p,q)}$$

and the modified Green's function as

$$(4)\quad \gamma_1(p,q) := \gamma_o(p,q) + \sum_{n=0}^{\infty}\sum_{m=0}^{n}[a_{mn}c^e_{mm}(p)c^e_{nm}(q) + b_{nm}s^e_{nm}(p)s^e_{nm}(q)].$$

The coefficients a_{mn} and b_{mn} in the *modification* are as yet arbitrary but will be chosen to eliminate interior resonances. It is vital to the ensuing formulation that the scatterer D_- have a non empty interior so that while the modification is singular at the origin (in D_-) it is regular in D_+ and on ∂D.

With j=0 or 1 denoting unmodified or modified Green's function we define single and double layer potentials

$$\left.\begin{aligned}(5)\quad (S_jw)(p) &:= \int_{\partial D} w(q)\gamma_j(p,q)\,ds_q\\ (6)\quad (D_jw)(p) &:= \int_{\partial D} w(q)\frac{\partial\gamma_j}{\partial n_q}(p,q)\,ds_q\end{aligned}\right\}\quad p\in\mathbb{R}^3\setminus\{0\}$$

where $\frac{\partial}{\partial n_q}$ is the derivative in the direction of the outward normal to ∂D at q. Denote by K_j the boundary integral operator

$$(7)\quad (K_jw)(p) := \int_{\partial D} w(q)\frac{\partial\gamma_j}{\partial n_p}(p,q)\,ds_q\ ,\ p\in\partial D$$

with adjoint in $L_2(\partial D)$

$$(8)\quad (K_j^*w)(p) := \int_{\partial D} w(q)\frac{\partial\bar{\gamma}_j}{\partial n_q}(p,q)\,ds_q,\ p\in\partial D.$$

where a bar denotes complex conjugate. In terms of K_j the jump conditions for single and double layer distributions are unaffected by the modification and are

(9) $\frac{\partial}{\partial n_p^{\pm}} S_j w = \pm w + K_j w, \quad p \in \partial D$

and

(10) $\lim_{p \to \partial D_{\pm}} D_j w = \mp w + \bar{K}_j^* w, \quad p \in \partial D.$

where $\frac{\partial}{\partial n_p^{\pm}}$ and $\partial D_{\pm}$ denote limiting values from D_+ and D_-. In this notation, Green's theorem applied to solutions, u, of the Helmholtz equation which satisfy a radiation condition at infinity yields the representation

(11) $(S_j \frac{\partial u}{\partial n})(p) - (D_j u)(p) = \begin{matrix} 2u, & p \in D_+ \\ u, & p \in \partial D \end{matrix}$

This in turn gives rise to a pair of boundary integral equations, one directly, and one by taking the normal derivative from the exterior and using the jump condition (9):

$$\left.\begin{aligned} &(12) \quad S_j \frac{\partial u}{\partial n} - \bar{K}_j^* u = u \\ &(13) \quad K_j \frac{\partial u}{\partial u} - \frac{\partial}{\partial n} D_j u = \frac{\partial u}{\partial n} \end{aligned}\right\} p \in \partial D.$$

These boundary integral relations give rise to boundary integral equations for Dirichlet and Neumann problems as follows:

2. DIRICHLET PROBLEM: $u = f, \ p \in \partial D$

Substituting the boundary data in (12) and (13) yields the boundary integral equations

(14) $(I - K_j) \frac{\partial u}{\partial n} = - \frac{\partial}{\partial n} D_j f.$

(15) $S_j \frac{\partial u}{\partial n} = (I + \bar{K}_j^*) f$

Alternatively one may assume a solution in the form of a double layer with unknown density

(16) $u = -D_j w, \quad p \in D_+$

which, with the jump relation and the boundary condition yields the boundary integral equation

(17) $(I - \bar{K}_j^*) w = f \quad p \in \partial D.$

3. NEUMANN PROBLEM: $\frac{\partial u}{\partial n} = g, \quad p \in \partial D$

Substituting the boundary data in (12) and (13) yields the boundary integral equations

(18) $(I + \bar{K}_j^*) u = S_j \, g$

(19) $\frac{\partial}{\partial n} D_j u = -(I - K_j) g.$

Alternatively the single layer ansatz

(20) $u = S_j w, \quad p \epsilon D_+$

with the jump relation (9) and the boundary condition yields the boundary integral equation

(21) $(I+K_j)w=g, \quad p \epsilon \partial D.$

In [6.47] it is shown that for j=0 the pair of equations (14) and (15) as well as the pair (18) and (19) have unique solutions thus providing unique solutions to the Dirichlet and Neumann problems for all real k. However if only the second kind equations are considered (14) and (18), or the layer equations (17) and (21), it is well known that these equations are not always uniquely solvable. In particular

(22) $(I-K_o)\frac{\partial u}{\partial n} = 0$ and $(I-\overline{K}_o^*)\ w = 0$

have non trivial solutions when k is an eigenvalue of the interior Neumann problem while

(23) $(I+K_o)\ w = 0$ and $(I+\overline{K}_o^*)\ u = 0$

have non trivial solutions when k is an eigenvalue of the interior Dirichlet problem.

It should be noted these values of k are real. On the other hand these equations also have non trivial solutions for sets of complex values of k which correspond to the SEM poles for the particular problem.

While the non uniqueness for real k can be removed by considering a pair of equations as noted above, an alternative resolution was proposed by Jones by means of modifying the Green's function. A modified form of Jones results is contained in the following (see [6.46]).

Theorem: If the coefficients of the modified Green's function satisfy the relations.

(24) $|2a_{nm}+1|<1$ and $|2b_{nm}+1|<1$

then

(25) $(I\pm K_1)\ w=0$ and $(I\pm\overline{K}_1^*)\ w=0$

have only the trivial solution for all real k. The theorem remains valid if the inequalities in (24) are reversed for all n and m. This means that by modifying the Green's function subject to (24), the equations of the second kind, (14), (17), (18), and (21) are uniquely solvable when j=1.

4. THE SPHERE

To shed some light on the way in which the modification has altered the location of the unwanted exceptional values of k and to see how the modification affects the SEM poles we examine the sphere case in detail.

First of all it is convenient to write the eigenvalues equation for K_j as

(26) $(I - \lambda(k)K_j)\ w=0$

where the eigenvalues λ will be functions of k and we are concerned with those values of k for which $\lambda(k) = \pm 1$ (+ for the exterior Dirichlet problem and - for the exterior Neumann problem.)

Because the spherical harmonics $\{P_n^m(\cos\theta)\begin{smallmatrix}\cos m\phi\\ \sin m\phi\end{smallmatrix}\}$ are complete in L_2 on the surface of the sphere it is a straight forward matter to compute the eigenvalues explicitly [e.g. [6.48] for the case j=0]. Using the notation (1), (2), the standard expansion of the free space Green's function is

(27) $$\gamma_o(p,q) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}[c_{nm}^e(p>)c_{nm}^i(p<)+s_{nm}^e(p>)s_{nm}^i(p<)]$$

where p< = p or q depending on which is the smaller of $\{r_p, r_q\}$ and p> is p or q depending on which is the larger. Also $c_{nm}^i(p)$ and $s_{nm}^i(p)$ are the same as $c_{nm}^e(p)$ and $s_{nm}^e(p)$ except that the spherical Bessel function $j_n(kr_p)$ replaces the spherical Hankel function $h_n^{(1)}(kr_p)$. Thus the eigenvalue equation is

(28) $$0=(I-\lambda(k)K_o)w=w(p)-\lambda(k)\int_o^{\pi}d\theta_p\int_o^{2\pi}d\phi_p w\sin\theta_p\frac{ka^2}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}$$
$$[c_{nm}^{e'}(p)c_{nm}^i(q)+c_{nm}^{i'}(p)c_{nm}^e(q)+s_{nm}^{e'}(p)s_{nm}^i(q)+s_{nm}^{i'}(p)s_{nm}^e(q)]$$

where ' denotes differentiation with respect to kr_p. Let $w = P_r^s(\cos\theta)\{\begin{smallmatrix}\cos s\phi\\ \sin s\phi\end{smallmatrix}\}$ and use the orthogonality of spherical harmonics to determine that

(29)
$$P_n^m(\cos\theta)\{\begin{smallmatrix}\cos m\phi\\ \sin m\phi\end{smallmatrix}\}[1+i\lambda(k)(ka)^2(h_n^{(1)'}(ka)j_n(ka)+h_n^{(1)}(ka)j_n'(ka))]=0$$

and hence

(30) $$\lambda_n(k) = -\frac{1}{i(ka)^2[h_n^{(1)'}(ka)j_n(ka)+h_n^{(1)}(ka)j_n'(ka)]}$$

is an eigenvalue of K_o of multiplicity 2n+1. Using the Wronskian relation

(31) $$j_n(ka)h_n^{(1)'}(ka)-j_n'(ka)h_n^{(1)}(ka) = \frac{i}{(ka)^2}$$

the eigenvalues may be written as

(32) $$\lambda_n(k)=-\frac{1}{1+2i(ka)^2j_n(ka)h_n^{(1)'}(ka)} = \frac{1}{1-2i(ka)^2j_n'(ka)h_n^{(1)}(ka)}$$

from which it is evident that the exceptional values of k for the Dirichlet problem, i.e. those values for which $\lambda_n(k) = 1$ are the zeros of $j_n'(ka)$ while the SEM poles are zeros of $h_n^{(1)}(ka)$. Similarly it is evident that the exceptional values of k for the Neumann problem, i.e. those values for which $\lambda_n(k) = -1$ are zeros of $j_n(ka)$ while the SEM poles are zeros of $h_n^{(1)'}(ka)$.

The calculation of the eigenvalues of the modified boundary integral operator K_1 is also easily carried out. Since

$$(33)\quad (I-\lambda)(k)K_1)w=w-\lambda(k)\int_0^{\pi}d\theta_p\int_0^{2\pi}d\phi_p\sin\theta_p a^2kw\sum_{n=0}^{\infty}\sum_{m=0}^{n}$$

$$\{\tfrac{1}{2}[c_{nm}^{e'}(p)c_{nm}^{i}(q)+c_{nm}^{i'}(p)c_{nm}^{e}(q)+s_{nm}^{e'}(p)s_{nm}^{i}(q)+s_{nm}^{i'}(p)s_{nm}^{e}(q)]$$

$$+\ a_{nm}c_{nm}^{e'}(p)c_{nm}^{e}(q)+b_{nm}s_{nm}^{e'}(p)s_{nm}^{e}(q)\}$$

we again set w equal to a particular spherical harmonic and utilize orthogonality to find

$$(34)\quad P_n^m(\cos\theta)\cos m\phi[1+i\lambda(k)(ka)^2(h_n^{(1)'}(ka)j_n(ka)+h_n^{(1)}(ka)j_n'(ka)$$

$$+\ 2a_{nm}h_n^{(1)'}(ka)h_n^{(1)}(ka))]=0$$

and

$$(35)\quad P_n^m(\cos\theta)\sin m\phi[1+i\lambda(k)(ka)^2(h_n^{(1)'}(ka)j_n(ka)+h_n^{(1)}(ka)j_n'(ka)$$

$$+2b_{nm}h_n^{(1)'}(ka)h_n^{(1)}(ka))]=0.$$

Hence the eigenvalues of K_1 are

$$(36)\quad \lambda(k)=-\frac{1}{i(ka)^2(h_n^{(1)'}(ka)j_n(ka)+h_n^{(1)}(ka)j_n'(ka)+2\alpha_{nm}h_n^{(1)'}(ka)h_n^{(1)}(ka))}$$

which may be rewritten using the Wronskian (31) as

$$(37)\quad \lambda(k)=-\frac{1}{1+2i(ka)^2h_n^{(1)'}(ka)(j_n(ka)+\alpha_{nm}h_n^{(1)}(ka))}$$

$$=\frac{1}{1-2i(ka)^2h_n^{(1)}(ka)(j_n'(ka)+\alpha_{nm}h_n^{(1)'}(ka))}\ .$$

where α_{nm} is either a_{nm} or b_{nm}. From (37) it is clear that the exceptional values of k for which $\lambda=1$ (Dirichlet problem) are roots of

$$(38)\quad (a)\ h_n^{(1)}(ka)=0 \quad\text{and}\quad (b)\ j_n'(ka)+\alpha_{nm}h_n^{(1)'}(ka)=0$$

whereas the exceptional values for $\lambda=-1$ (Neumann problem) are roots of

$$(39)\quad (a)\ h_n^{(1)'}(ka)=0 \quad\text{and}\quad (b)\ j_n(ka)+\alpha_{nm}h_n^{(1)}(ka)=0.$$

These results are summarized in the following table:

Exceptional values of k: $\lambda(k) = \pm 1$		
$\lambda(k) = 1$	Exterior Dirichlet Problem $(I - K_j)\, w = 0$	
$j = 0$	$j_n'(ka) = 0$	$h_n^{(1)}(ka) = 0$
$j = 1$	$j_n'(ka)+\alpha_{nm}h_n^{(1)'}(ka) = 0$	$h_n^{(1)}(ka) = 0$
$\lambda(k) = -1$	Exterior Neumann Problem $(I + K_j)\, w = 0$	
$j = 0$	$j_n(ka) = 0$	$h_n^{(1)'}(ka) = 0$
$j = 1$	$j_n(ka)+\alpha_{nm}h_n^{(1)}(ka) = 0$	$h_n^{(1)'}(ka) = 0$

From the chart it is evident that the effect of the modification of the fundamental solution is to move the eigenvalues (zeros) of j_n or j_n') off the real axis however the SEM poles (zeros of $h_n^{(1)'}$) remain unchanged.

The relation between the coefficients of the modification and the location of the shifted interior eigenvalues may be clarified by the following consideration. If 39(b) holds then

$$(40) \quad \alpha_{nm} = -\frac{j_n(ka)}{h_n^{(1)}(ka)}$$

and

$$(41) \quad 2\alpha_{nm}+1 = -\frac{2j_n(ka)}{h_n^{(1)}(ka)} + 1 = -\frac{h_n^{(2)}(ka)}{h_n^{(1)}(ka)} .$$

Let

$$w = 2\alpha_{nm}+1 \quad \text{and } z = ka$$

and consider the analytic transformation of conformal mapping

$$(42) \quad w = -\frac{h_n^{(2)}(z)}{h_n^{(1)}(z)} .$$

If z is real, $|w| = \left|\frac{h_n^{(2)}(z)}{h_n^{(1)}(z)}\right| = 1$ hence the real axis in the z-plane is mapped onto the unit circle in the w-plane. Since the zeros of $h_n^{(1)}(z)$ lie in the lower half plane* the lower half z-plane is mapped onto the exterior of the unit circle while the upper half plane is mapped onto the interior. From this it follows that if 39(b) holds than

$$\text{(43)} \quad \begin{aligned} &\text{a)} \quad |2\alpha_{nm}+1| < 1 \iff \operatorname{Im} ka > 0 \\ &\text{b)} \quad |2\alpha_{nm}+1| = 1 \iff \operatorname{Im} ka = 0 \\ &\text{and} \\ &\text{c)} \quad |2\alpha_{nm}+1| > 1 \iff \operatorname{Im} ka < 0 \end{aligned}$$

If (38b) holds then a similar consideration of the transformation

$$\text{(44)} \quad w = -\frac{h_n^{(2)'}(z)}{h_n^{(1)'}(z)}$$

shows that equations (43) remain valid.

We observe that the interior eigenvalues may be shifted to any point in the complex plane. That is if k_o is an arbitrary complex number it will be a zero of (38b) or 39(b) by choosing α_{nm} to be either

$$\text{(45)} \quad \alpha_{nm} = -\frac{j_n'(k_o a)}{h_n^{(1)'}(k_o a)} \quad \text{or} \quad \alpha_{nm} = -\frac{j_n(k_o a)}{h_n^{(1)}(k_o a)} .$$

However by choosing the constants α_{nm} such that 43a) holds we guarantee that the interior eigenvalues will be shifted into the upper half plane.

2. CONCLUDING REMARKS

The example considered shows that the boundary integral equations of the 2nd kind for exterior scattering problems can be modified by appropriate choice of the fundamental solutions so that they are uniquely solvable for all real k. Moreover there is considerable latitude in the choice of modified Green's function so that the unwanted interior eigenvalues may be shifted any place in the complex plane. The SEM poles, which are intrinsic to the exterior scattering problem are unaffected by the modification of the Green's function. Whether these statements all remain true for arbitrary nonspherical surfaces is not yet known. Certainly it is true that the modified integral equations are uniquely solvable for all real k for arbitrary smooth closed connected scatterers. Moreover it seems reasonable to conjecture that the interior eigenvalues are shifted for nonspherical surfaces just as they are for the sphere. Certainly they no longer are real and continuity arguments would indicate

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* See e.g. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards, U.S. Government Printing Office, 1964.

that small perturbations from spheres cause small shifts in the location of the shifted values. However the conjecture that the SEM poles are unaffected by the modification in the Green's function for nonspherical surfaces must be established in another way and this remains to be done.

NEW RELATIONS FOR THE CHARACTERISTIC SINGULARITIES OF BOUNDED SCATTERERS: PRELIMINARY REPORT

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1. INTRODUCTION

This paper deals with methods that are being developed to calculate the characteristic SEM singularities of perfectly conducting bodies - methods which seem to be different from the standard techniques used by the SEM community. The idea is to calculate a set of test functions with respect to which the existence of a certain orthogonal function is a necessary and sufficient condition that a certain complex number, ζ, is a singular value. These test functions are defined in terms of the eigen functions of the interior problem. Although the theoretical results do not depend upon the interior problem being separable, it is only in this case that it is possible to represent the functions in terms of the standard functions of mathematical physics.

Sections 2, 3, and 4 consist of a statement of these theoretical results and an outline of their proof. For the sake of simple exposition, the results will be stated in terms of scalar theory. This work is complete and has been extended to the electromagnetic case.

Section 5 is a status report on the attempt to use the results to determine the singularities for the exterior Dirichlet problem for the finite cylinder, an example, of course, of a problem for which the interior problem is separable. Although the work is incomplete, it illustrates a technique which other investigators may find useful. It consists of, using integral transform methods, finding representations which are only valid when $\mathrm{Im}\,\zeta < 0$, and then, by path deformation, obtaining a new representation which can be analytically continued into the lower half plane.

2. STATEMENT OF THE THEORETICAL RESULTS

R1: For x, y two points in space and ζ a complex number, $\mathrm{Im}\,\zeta < 0$, let $G(x,y;\zeta) = |x - y|^{-1} \exp(i\zeta|x - y|)$. Let B be the boundary, piecewise smooth, of a simple closed bounded region in space. A function h, defined on the interior region, is said to be trivial or nontrivial according to whether or not the function

$$V(x) = \iiint_{\mathrm{Int}} G(x,y;\zeta)\, h(y)\, d^3y$$

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is identically zero in the external region. If $\{v_k\}$ are the eigenfunctions for the interior Dirichlet problem, the test functions $\{w_k\}$ are defined, in the interior, as

$$w_k(x) = \iint_B G(x,y;\zeta) \frac{\partial \bar{v}_k(y)}{\partial n_y} \, dS(y)$$

Then the complex number ζ is a characteristic SEM singularity for the exterior Dirichlet problem if and only if there exists a nontrivial h for which

$$\iiint_{\mathrm{Int}} h(x) \, w_k(x) \, d^3x$$

is zero for all k. (Orthogonality condition)

To see how this works for the sphere, radius a, recall that the characteristic singularities are the values of ζ for which the spherical Hankel functions

$$h_n^{(1)}(\zeta a) = 0 \quad .$$

The eigenfunctions for the interior Dirichlet problem are

$$v_{mnl} = \exp(im\phi) \, P_n^{|m|} (\cos\theta) \, j_n(\kappa_{nl} r) \quad ,$$

$0 \leq n$, $|m| \leq n$, $j_n(\kappa_{nl} a) = 0$. The function G can be represented as:

$$G((\theta,\phi,r);\,(\theta',\phi',r');\,\zeta) \sim$$

$$\sum_{n,m} c_{nm} \exp(im(\phi - \phi')) \, P_n^m(\cos\theta) \, P_n^m(\cos\theta') \, j_n(\zeta r_<) \, h_n^{(1)}(\zeta r_>) \quad ,$$

in which $r_<$ and $r_>$ have their conventional meanings as the smaller and the larger of r and r', and the c_{nm} are constants. Neglecting constant nonzero factors ($j_n'(\kappa_{nl} a) \neq 0$), the test functions are

$$w_{nml} = \exp(-im\phi) \, P_n^m(\cos\theta) \, j_n(\zeta r) \, h_n^{(1)}(\zeta a) \quad .$$

If $h_{n_0}^{(1)}(\zeta a) = 0$ then, in virtue of the orthogonality of the zonal harmonics, $h = \exp(-im\phi) \, P_{n_0}^m(\cos\theta) \, j_{n_0}(\bar{\zeta} r)$ satisfies the orthogonality condition with respect to all $w^0_{mm\ell}$ for which $n \neq n_0$; since $w_{n_0 m\ell} = 0$, h satisfies the orthogonality condition. If V is computed, using this h, $V = \exp(im\phi) \, P_{n_0}^m(\cos\theta) \, h_{n_0}^{(1)}(\zeta r)$; therefore, h is not trivial. On the other hand, if $h_n^{(1)}(\zeta a) \neq 0$, all n, any h which satisfies the orthogonality condition must be orthogonal to all zonal harmonics and is therefore trivial.

R1 is a consequence of the following two results:

R2: For any h defined on the interior, the orthogonality condition is satisfied if and only if $V(x) = 0$ for all x on B.

R3: A complex number, ζ, Im $\zeta < 0$, is a characteristic SEM singularity if and only if there is a non trivial h, defined on the interior, for which $V(x) = 0$ for all x on B.

Remarks: For arbitrary h, defined on the interior, V(x), in the exterior region, is a solution of the scalar wave equation which satisfies the radiation condition, and whose sources are in the interior region. R3 states that if a V(x), with this representation, is not identically zero in the exterior region, and if $V(x) = 0$ on the boundary, then ζ must be a characteristic singularity; and, if ζ is a characteristic singularity, there must be a V(x), zero on the boundary, with such a representation, which is not identically zero in the exterior region. If the characteristic singularities are identified with the singularities of the analytic continuation of the resolvent Green's function, the first statement follows from the Lax Phillips Theory [6.52]. As will be seen below, the proof given for R3, makes use of the identification of the characteristic singularities with the singularities which occur in the integral equation (of the second kind) formulation of the scattering problem [6.26].

R2 states that a V(x), in this form, not identically zero in the exterior region, but for which $V(x) = 0$ on the boundary, exists if and only if a source function can be found which satisfies the orthogonality condition. R2 is the substantially novel result of this paper.

3. THE PROOF OF R2

Let $\{v_k\}$ be the complete orthonormal set of eigenfunctions for the interior Dirichlet problem; the eigenvalues, λ_k^2, are real and positive. Since

$$\nabla^2 v_k + \lambda_k^2 v_k = 0 \ , \ v_k = 0 \text{ on } B \ ,$$

and

$$\nabla^2 G + \zeta^2 G = 0 (x \neq y) \ ,$$

$$(G, v_k) = \iiint_{Int} G(x,y;\zeta)\, \bar{v}_k(x)\, d^3x$$

can be calculated in the conventional way to be

$$(\zeta^2 - \lambda_k^2)^{-1} \left\{ \iint_B G(x,y;\zeta) \frac{\partial \bar{v}_k(x)}{\partial n_x} dS(x) - 4\pi \bar{v}_k(y) \right\}$$

$$= (\zeta^2 - \lambda_k^2)^{-1} \{ w_k(y) - 4\pi \bar{v}_k(y) \} \ ,$$

whenever y is an interior point.

Suppose, first, that $V(x) = 0$ for all x on B. Then V(x) is the solution to e interior Dirichlet problem

$$\nabla^2 V + \zeta^2 V = -4\pi h \ .$$

As usual, the generalized Fourier coefficients are

$$(V,v_k) = -4\pi(\zeta^2 - \lambda_k^2)^{-1}\,(h,v_k) \quad . \qquad \ldots.(*)$$

On the other hand,

$$(V,v_k) = \iiint_{Int} \left\{ \iiint_{Int} G(x,y;\zeta)\; h(y)\; d^3y \right\} \bar{v}_k(x)\; d^3x$$

$$= \iiint_{Int} (G,v_k)\; h(y)\; d^3y$$

$$= (\zeta^2 - \lambda_y^2)^{-1} \iiint_{Int} \{w_k(y) - 4\pi\bar{v}_k(y)\}\; h(y)\; d^3y$$

Thus,

$$(V,v_k) = (\zeta^2 - \lambda_k^2)^{-1} \left\{ \iiint_{Int} h(x)\; w_k(x)\; d^3x - 4\pi(h,v_k) \right\} \qquad \ldots (**)$$

If this representation is compared to (*), it is clear that h satisfies the orthogonality condition.

Conversely, if the orthogonality condition holds, (**), which depends only on the definition of V, shows that (V,v_k) is given by (*).

If the derivation of (*) is reviewed, it is apparent that V has the same Fourier coefficients as the solution to the Dirichlet problem; since $\{v_k\}$ are complete and both V and the solution to the Dirichlet problem are continuous, they are identical.

The proof is complete.

4. IDENTIFICATION OF THE CHARACTERISTIC SEM SINGULARITIES AND THE PROOF OF R3

The inversion of the operator

$$[T\phi](y) = \phi(y) + (2\pi)^{-1} \iint_B \frac{\partial}{\partial n_y} G(x,y;\zeta)\; \phi(x)\; dS(x) \quad ,$$

when $\mathrm{Im}\,\zeta > 0$, is equivalent to calculating the resolvent Green's function for the exterior Dirichlet problem; the operator is, in fact, invertible for $\mathrm{Im}\,\zeta > 0$. The characteristic SEM singularities are those values of ζ, $\mathrm{Im}\,\zeta < 0$, for which the operator is not invertible, or, equivalently, the singularities of the analytic continuation of the resolvent Green's function into the lower half plane. From the Fredholm alternative, the singularities are precisely the values of ζ , $\mathrm{Im}\,\zeta < 0$, for which the adjoint problem

$$[T^*\bar{\phi}](y) = \bar{\phi}(y) + (2\pi)^{-1} \iint_B \frac{\partial}{\partial n_x}\; G(x,y;-\bar{\zeta})\; \bar{\phi}(x)\; dS(x) = 0$$

has solutions not identically zero. By taking complex conjugates, this is equivalent to the *equation*

$$\phi(y) + (2\pi)^{-1} \iint_B \frac{\partial}{\partial n_x} G(x,y;\zeta)\, \phi(x)\, dS(x) = 0 \;.....\; (I)$$

having nonzero solutions. R3 is, therefore, an assertion that for $\operatorname{Im}\zeta < 0$, this integral equation has a nonzero solution if and only if there exists an h, defined on the interior, which is nontrivial, and for which

$$V(x) = \iiint_{Int} G(x,y;\zeta)\, h(y)\, d^3y$$

is zero when x is on B.

Before proceding with the proof, it should be observed, first, that the characteristic singularities occur in pairs, ζ and $-\bar{\zeta}$; and second, that it is known, from the Lax-Phillips theory, for arbitrary nontrivial h, ζ is a characteristic singularity whenever V(x) is identically zero on B, because

$$\nabla^2 V + \zeta^2 V = 0 \quad ,$$

in the exterior region and

$$V \sim \exp(i\zeta|x|), \text{ as } |x| \to \infty \quad .$$

The first half of the proof of R3 is a restatement of this fact.

Assume that h is nontrivial and that V(x) = 0 on B. Let u be the unique solution of the interior Neumann problem.

$$\nabla^2 u + \zeta^2 u = -4\pi h, \quad \frac{\partial u}{\partial n} = 0 \text{ on } B.$$

Then, as is standard, for all y in the interior,

$$u(y) + (4\pi)^{-1} \iint_B \frac{\partial}{\partial n_x} G(x,y;\zeta)\, u(x)\, dS(x) = V(y) \;.......(\#)$$

and for all y on B,

$$u(y) + (2\pi)^{-1} \iint_B \frac{\partial}{\partial n_x} G(x,y;\ )\, u(x)\, dS(x) = 2V(y) = 0 \;...(\#\#)$$

On B, therefore, u is a solution of (I). To prove that ζ is a singularity it is sufficient to show that u is not identically zero on B. If it were, from (#), u and V would coincide on the interior, as well as on B. Therefore, both V and its normal derivative vanish at B. From this, and the form of V, it can be shown that V is identically zero in the exterior region. Consequently, h is trivial, contrary to hypothesis.

To prove the converse, assume that ϕ is a nonzero solution of the integral equation (I). It can be shown that functions exist, twice differentiable on the interior, with normal derivative zero at B, and having the value ϕ on B. Let u be such a function. Define h, in the interior, as

$$h = -(4\pi)^{-1}\{\nabla^2 u + \jmath^2 u\}.$$

Since u, on the boundary, is a solution of (I), V, for this h, is zero on B. It remains to show that h is nontrivial. If not, V would have zero normal derivative at B, and therefore V and u would be solutions of the same interior Neumann problem. Thus V = u, implying the u is zero on B. This is a contradiction.

The proof is complete.

5. THE APPLICATION OF R1

In order to use R1, it is required that the eigenfunctions $\{v_k\}$ be computed, that a representation for $G(x,y;\jmath)$ be found in a form which permits the calculation of the $\{w_k\}$, and that the orthogonality conditions be used to make inferences about the singularities. For the sphere, a fortuitous confluence of the representations for $\{v_k\}$ and G, in terms of zonal harmonics, makes the process simple. (Perhaps it should be noted that the argument in Section 1 is a very simple demonstration that <u>only</u> the complex roots of $h_n^{(1)}(\jmath a)$ are characteristic singularities.)

In general, even for the case in which the interior Dirichlet problem is separable (e.g.: hemisphere, finite circular cylinder) the problem seems to be very difficult. However, one has the advantage that the $\{v_k\}$ are easy to calculate in terms of well known functions. To obtain the w_k in closed form the following steps are taken:

First, classical integral transforms are used to obtain representations for G. Because these transforms require that the functions to which they are applied be very small for large values of the argument, this calculation can only be performed when $\mathrm{Im}\,\jmath > 0$.

Second, the w_k are computed, obtaining a closed form representation as an integral.

Third, the integral is transformed by a deformation of path; a new closed form representation is obtained. It can be continued into the lower half plane.

As an example, let the surface be that of a finite circular cylinder, with radius a and length L. The interior region is defined by cylindrical coordinates

$$0 \le r \le a,\ 0 \le z \le L,\ \text{and } 0 \le \phi \le 2\pi\ .$$

The eigenfunctions of the interior Dirichlet problem are

$$v_{nkm} = \exp(in\phi)\ \sin\frac{k\pi z}{L}\ J_{|n|}(\gamma_{nm}r),\quad J_n(\gamma_{nm}a) = 0.$$

The function $G = G((r,\phi,z),\ (r',\phi',z');\mathfrak{z})$ is the solution of

$$\nabla^2 G + \mathfrak{z}^2 G = -4\pi\ \delta(r,\phi,z;r',\phi',z')$$

which is regular at $r = 0$, and which $\sim \exp(i\mathfrak{z}|x|)$ when $|x|$ is large. ∇^2 is expressed of course in cylindrical coordinates.

Then the test functions are:

$$w_{nkm} = \exp(-in\phi)\Bigg\{a\ \gamma_{nm}\ J'_{|n|}(\gamma_{nm}a)\int_0^L \sin\frac{k\pi z'}{L}\ U_n(r,z;a,z')\ dz'$$

$$+\frac{k\pi}{L}\int_0^a J_{|n|}(\gamma_{nm}r')\ (-1)^k\ U_n(r,z;r',L) - U_n(r,z;r',0)\quad r'\ dr'\Bigg\}\ \ldots\ldots(w)$$

where

$$\frac{\partial^2}{\partial r^2}U_n + \frac{1}{r}\frac{\partial}{\partial_r}U_n + (\mathfrak{z}^2 - n^2/r^2)\ U_n + \frac{\partial^2}{\partial z^2}U_n = -4\pi\ \delta(r,z;r',z')\ .$$

The Fourier transform is applied to this differential equation, the transform being taken with respect to z, using s as the transform variable. Using standard methods, the transform U_n^* is found:

$$U_n^*(r,s;a,z') = -i\pi\ J_{|n|}(pr)\ H_{|n|}^{(1)}(pa)\ \exp(-isz')\ ;$$

p is the branch of $(\mathfrak{z}^2 - s^2)^{\frac{1}{2}}$ whose imaginary part is positive when s is real. U_n is then recovered from U_n^*, in the form of a Fourier integral. The first integral which appears in the formula (w) for w_{nkm} is

$$\int_0^L \sin\frac{k\pi z'}{L}\ U_n(r,z;a,z')\ dz' =$$

$$\int_{-\infty}^{\infty}\frac{k\pi^2}{L}\ J_{|n|}(pr)\ H_{|n|}^{(1)}\ (pa)\left[\frac{(-1)^k\ \exp(-isL) - 1}{s^2 - k^2\pi^2/L^2}\right]\exp(isz)\ ds\ .$$

To calculate the second integral which appears in (w), the process is repeated using a Fourier-Bessel transform with respect to r, and the transform variable t. Suppressing a nonzero factor, the closed form representation for w_{nkm}, valid for $\mathrm{Im}\,\mathfrak{z} > 0$, is

$$w_{nkm} = \exp(-in\phi)\left\{\int_{-\infty}^{\infty} J_{|n|}(pr)\, H^{(1)}_{|n|}(pa)\left[\frac{(-1)^k \exp(-isL) - 1}{s^2 - k^2\pi^2/L^2}\right]\exp(isz)\, ds\right.$$

$$\left. + 2\int_{0}^{\infty}\left[\frac{(-1)^k \exp(iq(L - z)) - \exp(iqz)}{q}\right]\frac{J_{|n|}(tr)\, J_{|n|}(ta)}{t^2 - \gamma^2_{nm}}\, t\, dt\right\} \quad ;$$

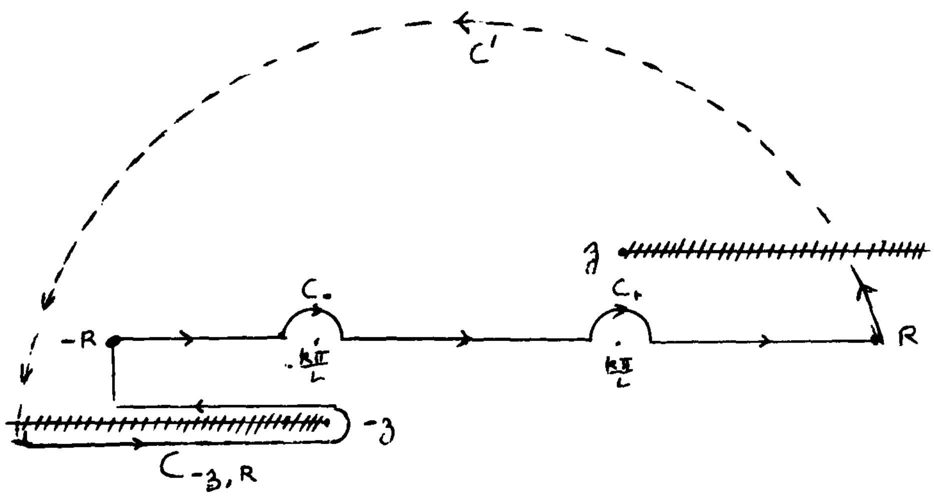
q is the branch of $(\mathfrak{z}^2 - t^2)^{\frac{1}{2}}$ whose imaginary part is postive when t is real.

The final step is to reevaluate these integrals in a form which can be continued into the lower half plane. To illustrate the method this is done for the first integral.

Designating the integral by I, $I = \lim I_e \cdot I_e$ is the integral along the real s axis, deleting an interval of width e at each of the points $s = k\pi/L$ and $s = -k\pi/L$. The limit is taken as e approaches zero.

Then the integrand is written as the sum of two terms one of which contains the factor $\exp(-is(L - z))$; the other contains the factor $\exp(isz)$. Attention is fixed on the term which contains the factor $\exp(isz)$. The notation I_e will be retained, for this term.

On the Riemann surface for $(\mathfrak{z}^2 - s^2)^{\frac{1}{2}}$, above the s plane, the following path is constructed:



I_R is the integral along the deleted path, but for $-R \leq s \leq R$. Clearly, I_e is the limit of I_R as R becomes large. The dotted line in the diagram corresponding to the arc C', is on the second sheet of the surface; the hatched lines are the cuts in the s plane. The indicated curve is simple and closed on the Riemann surface. Consequently,

$$I_R + I_{C_-} + I_{C_+} + I_{C'} + I_{C_{-\mathfrak{z},R}} = 0$$

It is then routine to verify that as R becomes arbitrarily large, $I_{C'}$ converges to zero, and that $I_{C_{-\zeta},R}$ can be written as an integral, along the upper edge of the cut, of a function in which only Bessel functions appear. I_{C_-} and I_{C_+} are easy to evaluate as e goes to zero.

A similar calculation can be performed for the other exponential factor. After some manipulation the integral can be written:

$$L/k \sin k\pi z/L \; J_{|n|}((\zeta^2 - k^2\pi^2/L^2)^{\frac{1}{2}} r) \; H^{(1)}_{|n|}((\zeta^2 - k^2\pi^2/L^2)^{\frac{1}{2}} a)$$

$$+ \int_{\zeta}^{\zeta+\infty} J_{|n|}(pr) \; J_{|n|}(pa) \left[\frac{(-1)^k \exp(-is(L - z)) - \exp(-isz)}{s^2 - k^2\tau^2/L^2}\right] ds \; .$$

The integrand is an analytic function of ζ, and the integral converges for all ζ ; the remaining terms have well known continuations into the lower half plane.

COMPLEX SINGULARITIES OF THE IMPEDANCE FUNCTIONS OF ANTENNAS

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I. INTRODUCTION

Most of the research on the singularity expansion method are based on the integral equation of the surface current on a conducting scatterer or the eigenfunction of the natural oscillation of some bodies of simple shapes. An alternative approach, more closely related to the transient response of antennas, is to study the complex singularities of the impedance function of these antennas. This approach is similar to the search of singularities for a terminated line [1]. In fact, for a biconical antenna the model is exactly the same because the input impedance of a biconical antenna can be interpreted as that of a terminated biconical transmission line [2,3]. For antennas of arbitrary shape they can be treated as nonuniform biconical antennas by a perturbation method originally due to Schelkunoff [3]. However, for this class of antennas the average characteristic impedance of the antenna, which is a parameter used in the perturbation theory, is arbitrary. This procedure is similar to that of the thickness parameter in higher order solutions of Hallen's theory based on the integral equation method.

In this paper we will first review the basic method used in the transmission line theory and then apply it to thin biconical antennas. Schelkunoff's perturbation method, with a proper modification of the value of the average characteristic impedance, is then applied to thin cylindrical and prolate spheroidal antennas.

II. ZEROS OF THE INPUT IMPEDANCE FUNCTION OF A TERMINATED LINE

For a lossless line terminated by a load impedance Z the normalized input impedance function $z_i(s)$ of the line expressed in Laplace transform domain is given by

$$z_i(s) = \frac{1 + \Gamma(s)e^{-2s}}{1 - \Gamma(s)e^{-2s}} \ , \tag{1}$$

where

$$s = j\left(\frac{\omega \ell}{c}\right) \ ,$$

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$$\Gamma(s) = \frac{Z(s) - Z_c}{Z(s) + Z_c} = \frac{Y_c - Y(s)}{Y_c + Y(s)}$$
$$= \frac{1 - y(s)}{1 + y(s)} \tag{2}$$

$$Z_c = \frac{1}{Y_c} = \text{characteristic impedance of the line.}$$

For a line termined by a series R-L-C impedance, for example, the zeroes of $z_i(s)$, denoted by s_n, are roots of the equation

$$1 + \Gamma(s)e^{-2s} = 0 \quad , \tag{3}$$

where
$$\Gamma(s) = \frac{z(s) - 1}{z(s) + 1} \quad , \quad z(s) = r + \alpha s + \frac{\beta}{s} \quad .$$

Three typical distributions oı $\cdot_n$ based on the solution of (3) are shown in Figs. 1 through 3.

III. ZEROS OF INPUT IMPEDANCE FUNCTION OF THIN BICONICAL ANTENNAS

According to the theory of biconical antennas [2,3] the input impedance function of these antennas can be written in the form of Eq. (1) except that the characteristic impedance is now replaced by that of the biconical transmission line and the load impedance or admittance by an effective terminated function resulting from the radiation of the antenna. For thin biconical antennas the characteristic impedance is given by

$$Z_c = \frac{Z_o}{\pi} \ln \frac{2}{\theta_o}$$

where

$$Z_o = (\mu_o/\varepsilon_o)^{1/2} \; ,$$

$$\theta_o = \text{half-angle of the bicone,}$$

and y(s), the effective normalized terminal admittance expressed in Laplace transform domain, has the expression

$$y(s) = \frac{Y(s)}{Y_c} = \frac{Z_o Y_c}{4} \{2L(2s) + e^{-2s}[\ln 2 + L(2s) - L(4s)] + e^{2s}[-\ln 2 + L(-2s)\} \quad , \tag{4}$$

where $s = j\omega\ell/c$,

ℓ = length of the biconical antenna,

$c = (\mu_o\varepsilon_o)^{-1/2}$,

$L(x) = \int_0^x (1-e^{-t})/t \; dt$.

The zeros of the input impedance function of a typical thin biconical antenna are shown in Fig. 4. They are first obtained numerically by a scanning search method and later verified by Giri based on a more systematic contour integration method [4]. It is quite certain that there are only two branches or layers in the upper left half plane in the s domain. There are, of course, two conjugate branches in the lower left half plane.

IV. ZEROS OF THE IMPEDANCE FUNCTIONS OF THIN CYLINDRICAL AND SPHEROIDAL ANTENNAS

According to Schelkunoff [2] antennas of arbitrary shape can be treated as nonuniform biconical antennas. By using a perturbation method it is possible to calculate the impedance of these antennas. However, the parameter, corresponding to the characteristic impedance of the uniform biconical antenna used in the perturbation method, is arbitrary. This parameter plays a similar role as the thickness parameter in Hallen's theory of cylindrical antennas based on the integral equation method. In Schelkunoff's original work he used the average characteristic impedance of these antennas as the expansion parameter. They are:

$$Z_{ca} = \begin{cases} \dfrac{Z_o}{\pi}\left(\ell n \dfrac{2\ell}{a} - 1\right) , & \text{for cylindrical antennas} \\[2ex] \dfrac{Z_o}{\pi}\, \ell n \dfrac{\ell}{a} , & \text{for prolate spheroidal antennas} \end{cases}$$

The biconical antennas with these characteristic impedances have the dimension shown in Fig. 5a by the dotted lines. The corresponding cylindrical antenna and the spheroidal antenna are shown in the same figure. The impedances of these antennas based on this model do not agree well with the results based on other methods, such as the one due to King and Middleton [5] and the variational solution [6]. We, therefore, have revised Schelkunoff's theory by using an average impedance

$$Z'_{ca} = \frac{Z_o}{\pi}\, \ell n \frac{2\ell}{a} .$$

The biconical antenna with this characteristic impedance is shown by the dotted line in Fig. 5b.

The values of the impedance calculated based on this modified parameter in the perturbation method agree much better with the results of other theories. Following Schelkunoff's theory the relevant formulas are:

$$z_i(s) = \frac{y(s)A(s) + B(s)}{y(s)C(s) + D(s)} , \tag{5}$$

where

$$\left.\begin{matrix} A(s) \\ D(s) \end{matrix}\right\} = -Sh(s) \pm \frac{N(s)Ch(s) + M(s)Sh(s)}{Z'_{ca}}$$

$$\left.\begin{matrix} B(s) \\ C(s) \end{matrix}\right\} = -Ch(s) \pm \frac{N(s)Sh(s) + M(s)Ch(s)}{Z'_{ca}}$$

$$N(s) = -\frac{Z_o}{2\pi}\left[Sh(2s) - \frac{1}{2}L(2s) + \frac{1}{2}L(-2s)\right] ,$$

$$M(s) = \frac{Z_o}{2\pi}\left[Ch(2s) - 1 + \frac{1}{2}L(2s) + \frac{1}{2}L(-2s)\right] .$$

The function y(s) is the same as the one given by Eq. (4) with Z_c therein replaced by Z'_{ca}. A typical distribution of the zeros of $z_i(s)$ for cylindrical antennas based on this method is shown in Fig. 6. The zeros of the corresponding inscribed biconical antennas are shown in the same figure. Figure 7 shows a comparison of these zeros for another cylindrical antenna with the results obtained by Tesche [7] based on the integral equation method. Figure 8 gives a comparison of our result for the prolate spheroidal antenna with the one obtained by Marin [8]. There is very little similarity between our calculated values and their findings. In particular, like the distribution of the zeros for thin biconical antennas there are only two distinct branches for both cylindrical and spheroidal antennas based on the present method, and there are more branches or layers according to Tesche's calculations. We are unable to offer an explanation of these differences.

V. POLES OF THE CURRENT RESPONSE FUNCTION OF A BICONICAL RECEIVING ANTENNA

According to the well known theory of receiving antennas the load current of an antenna terminated by a load impedance operated as a receiving antenna placed in an incident field is given by

$$I(s) = \frac{\bar{E}^i(s) \cdot \bar{h}(s)}{Z_L(s) + Z_i(s)} ,$$

where $\bar{E}^i(s)$ = incident electric field,

$\bar{h}(s)$ = vector effective height function,

$Z_i(s)$ = input impedance function of the antenna operating in its transmitting mode and

$Z_L(s)$ = terminal load impedance.

All quantities are defined in terms of the Laplace transform variable 's'. The zeros of $Z_L(s) + Z_i(s)$ correspond to the poles of the current response function. There may be other poles associated with the excitation function $\bar{E}^i(s)$ and the effective height function $\bar{h}(s)$ of the antenna which are not under discussion. Based on this model we have calculated the roots of the equation

$$Z_L(s) + Z_i(s) = 0$$

for a thin biconical antenna. A typical distribution of the poles of the current response function is shown in Fig. 9 for a resistive load. It is seen as the load resistance changes from zero (short circuit) to infinity (open circuit) the poles of I(s) start from the zeros of $Z_i(s)$, marked by dots, then migrate and terminate at the poles of $Z_i(s)$, marked by crosses. All poles are simple except the one located on the negative real axis at

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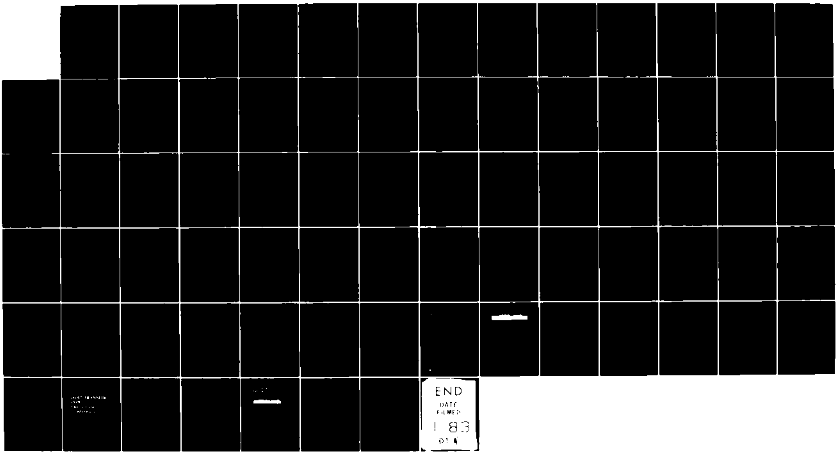

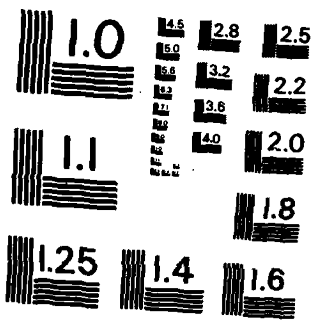

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$s_n = -1.61$. It is interesting to note that the poles started at the zeros of $Z_i(s)$ from the second branch recede to negative infinity for a matched load ($R_L = Z_c$ or $\alpha = 1$) but those started at the first branch migrate in the finite plane. The effect of loading as exhibited by this plot has not been studied by other workers based on the integral equation method. No comparison, therefore, can be made

VI. CONCLUSIONS

The zeros of the impedance function of thin biconical, cylindrical, and prolate spheroidal antennas have been calculated based on a method similar to the one for a terminated transmission line. For cylindrical and spheroidal antennas Schelkunoff's perturbation theory with a proper modification has been used. The results, in general, show considerable difference as compared with those obtained by the integral equation method. In particular, we have found only two distinct branches of zeros in contrast to many layers found by other workers. The distribution of the zeros based on the impedance method is very similar to that of a transmission line terminated by an impedance. It is possible that the difference could be due to different approximations involved in the two methods. As far as the time domain solution is concerned the singularities with low real damping constant are the significant ones for the transient response. Since the first branch based on the two different methods is very close the actual transient response may not be significantly different. The poles of the current response function of a biconical receiving antenna have also been investigated to illustrate the application of the present method for this class of problems involving a loaded antenna. A decent explanation requires a thorough examination of the uniqueness problem based on different methods. When an approximate formulation is involved as in our present work where the terminal admittance function is an approximate expression for thin biconical antenna it is not clear whether or not it may have a drastic effect on the distribution of its 'exact' singularities. The assistance of Mr. Soon K. Cho in the numerical computation is gratefully acknowledged.

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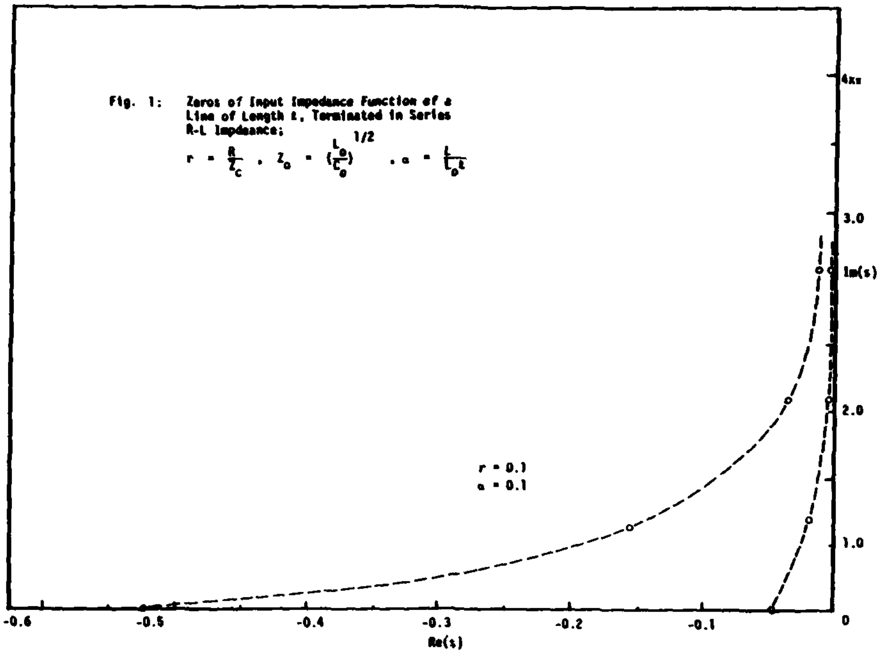


Fig. 1: Zeros of Input Impedance Function of a Line of Length ℓ, Terminated in Series R-L Impedance;
$r = \frac{R}{Z_c}$, $Z_o = (\frac{L_o}{C_o})^{1/2}$, $\alpha = \frac{L}{L_o \ell}$

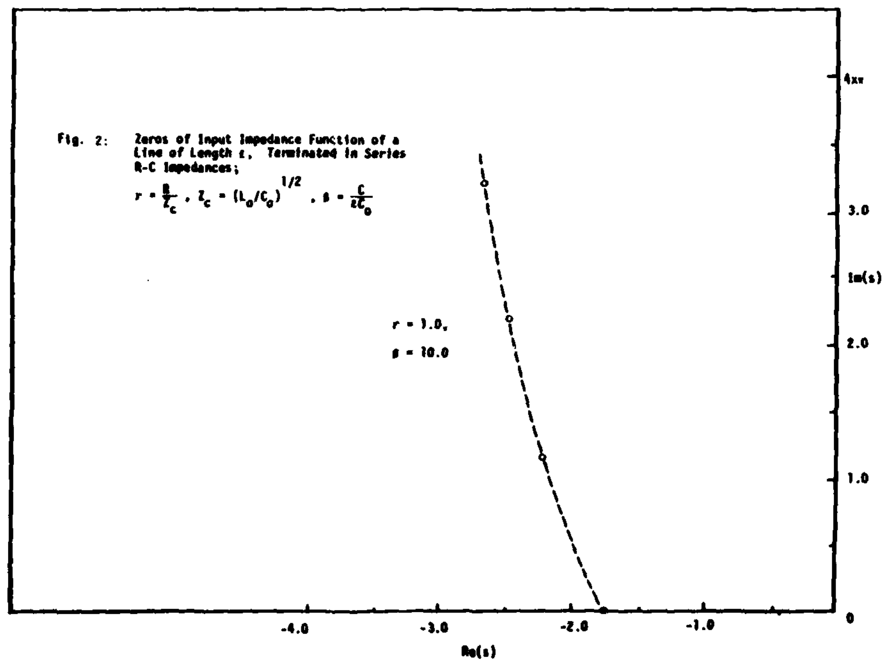


Fig. 2: Zeros of Input Impedance Function of a Line of Length ℓ, Terminated in Series R-C Impedances;
$r = \frac{R}{Z_c}$, $Z_c = (L_o/C_o)^{1/2}$, $\beta = \frac{C}{\ell C_o}$

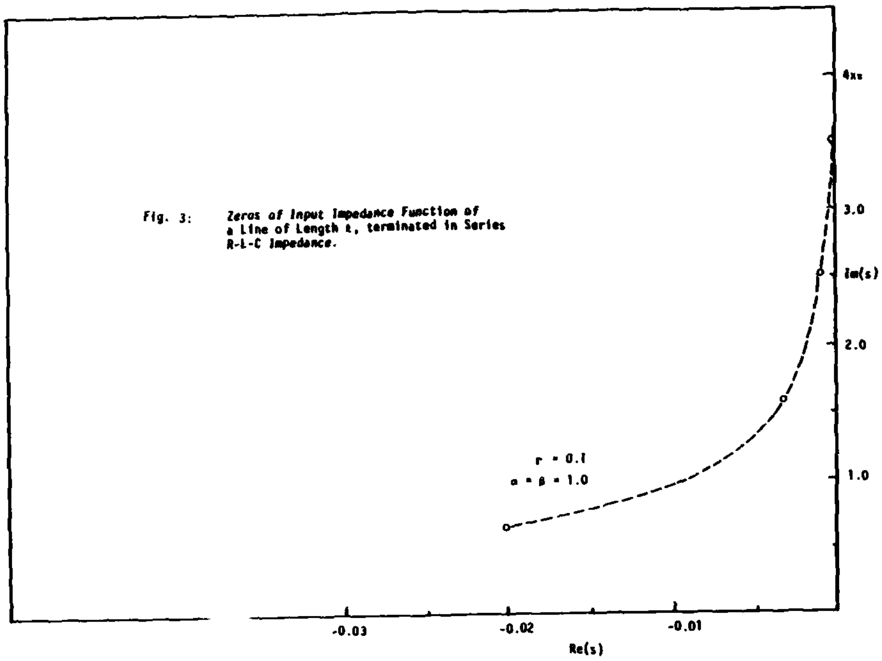


Fig. 3: Zeros of Input Impedance Function of a Line of Length ℓ, terminated in Series R-L-C Impedance.

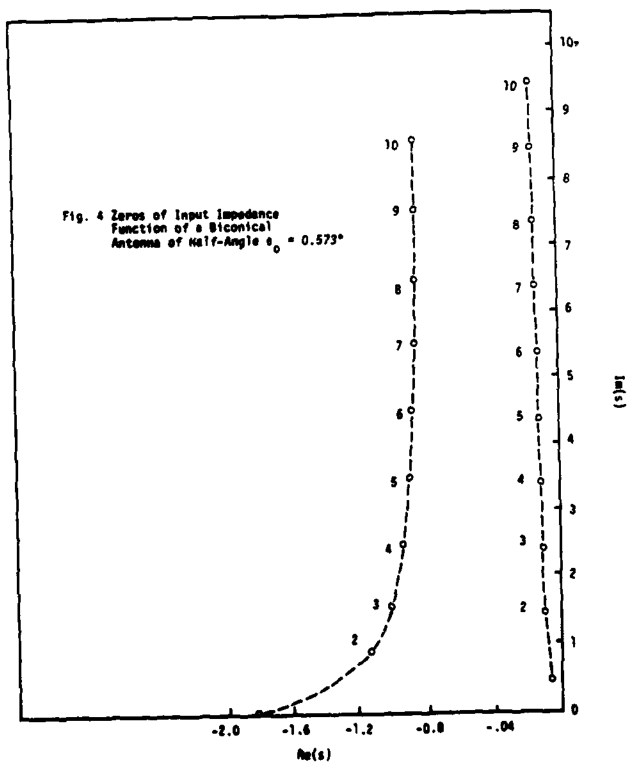
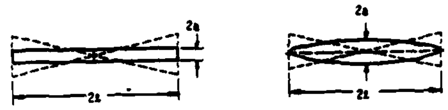


Fig. 4 Zeros of Input Impedance Function of a Biconical Antenna of Half-Angle θ_0 = 0.573°



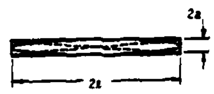
a) Perturbated cone with characteristic impedance equal to

$$Z_{ca} = \frac{Z_0}{\pi}\left(\ln\frac{2\ell}{a} - 1\right)$$

for cylinder and

$$Z_{ca} = \frac{Z_0}{\pi}\ln\frac{\ell}{a}$$

for a spheroid. (Schelkunoff)



b) Perturbated cone with characteristic impedance equal to

$$Z'_{ca} = \frac{Z_0}{\pi}\ln\frac{2\ell}{a}$$

Fig. 5: Perturbated cones and the corresponding cylindrical and spheroidal antennas.

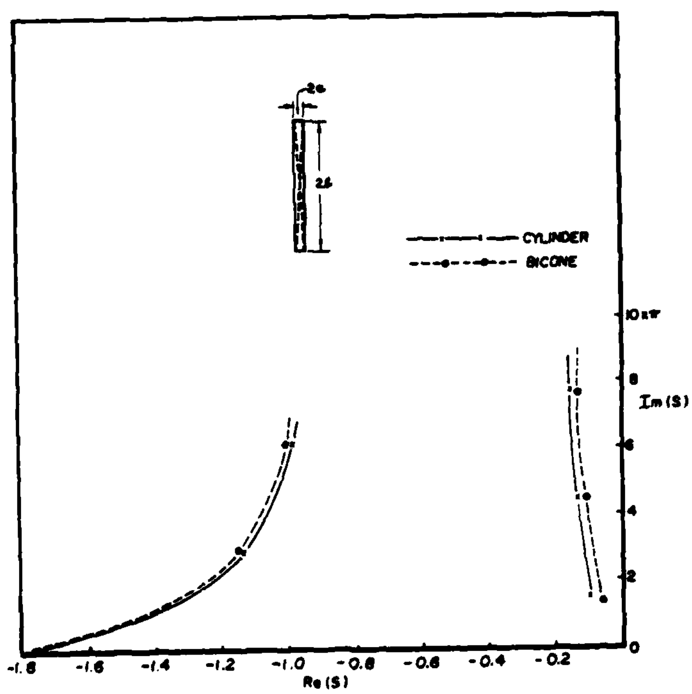


Figure 6: Comparison of the first three poles of input impedances for a cylindrical antenna and a biconical antenna for the parameter $2a/\ell = 0.1$.

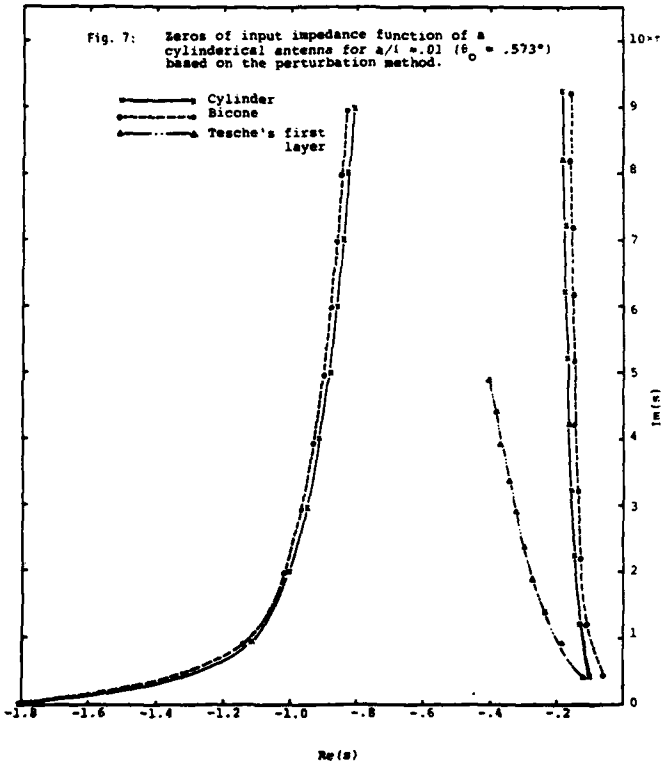


Fig. 7: Zeros of input impedance function of a cylinderical antenna for $a/\ell = .01$ ($\theta_o = .573°$) based on the perturbation method.

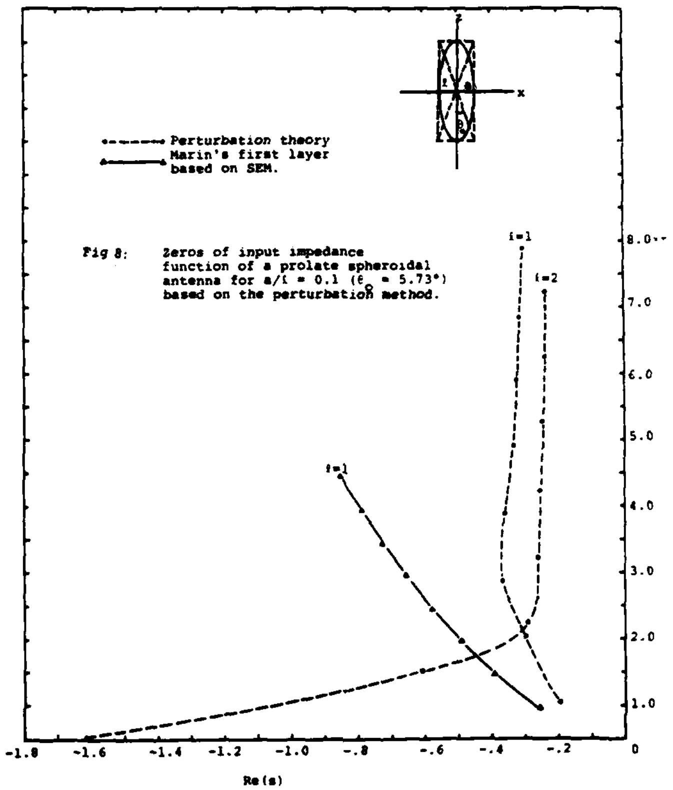


Fig 8: Zeros of input impedance function of a prolate spheroidal antenna for $a/\ell = 0.1$ ($\theta_0 = 5.73°$) based on the perturbation method.

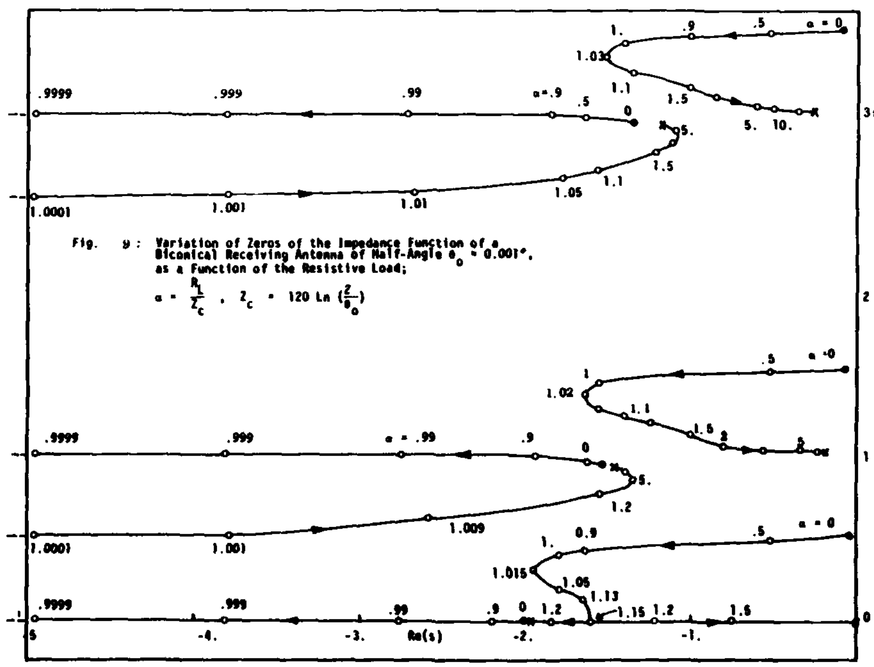


Fig. 9: Variation of Zeros of the Impedance Function of a Biconical Receiving Antenna of Half-Angle $\theta_0 = 0.001°$, as a Function of the Resistive Load;

$\alpha = \frac{R_L}{Z_c}$, $Z_c = 120 \ln (\frac{Z}{\theta_0})$

ON THE USE OF SINGULARITY EXPANSION METHOD FOR ANALYSIS OF ANTENNAS IN CONDUCTING MEDIA

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ABSTRACT

The application of the singularity expansion method (SEM) to the analysis of antennas and electromagnetic scatterers has usually been applied to simple, isolated bodies in free space or to simple bodies near a perfectly conducting ground plane. Theoretical studies of SEM have been applied to these relatively simple geometries to yield significant insight into the radiation and scattering process. In analytically investigating the behavior or antennas in a lossy medium, it is known that in addition to simple pole singularities, there is a branch cut linking two branch points in the complex frequency representation of the antenna response. While significant information regarding the nature of the branch cut and its effect on the antenna response can be obtained by purely analytical methods, a numerical study of this antenna can provide useful results.

This paper provides an analysis of the behavior of a linear antenna in a conducting region. Special attention is paid to the importance of the branch cut contribution to the overall antenna response. The possibility of realizing the input admittance or impedance of the antenna via lumped circuit elements is first reviewed for the case of a lossless medium surrounding the antenna, and then extended to the case of an antenna in a lossy medium.

1. INTRODUCTION

The introduction of the singularity expansion method (SEM) by Baum [3.1] has resulted in a useful method for determining the wide band or transient electrical behavior of antennas and scatterers. Due to the mathematical complexity of SEM, however, it is not generally possible to perform a thorough analysis on such problems without resorting to numerical methods. Even in the case of an extremely simple body, say a sphere, numerical methods are needed to evaluate the complex singularities (poles) of the response. Furthermore, significant theoretical questions regarding SEM, such as completeness and the existence of entire functions, remain unanswered.

Whereas numerical calculations can be used to examine some of these unanswered theoretical questions in SEM, they certainly cannot be considered to be a "proof" of a particular fact. They can, however, serve to guide the theoretical studies of SEM by providing data on selected geometrical configurations. This paper presents results of a short study of a linear antenna immersed in a conducting medium. Of particular interest are:

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(i) the computation of input admittance Y_{in} and short circuit current I_{sc} via a knowledge of the complex singularities and (ii) one form of synthesizing these quantities using lumped parameter networks (LPNs).

This paper is organized as follows. After this introductory section, Section 2 briefly reviews the SEM analysis of the problem of cylindrical antennas in free space. In Section 3, the SEM analysis of a cylindrical antenna in a conducting medium is considered and methods of synthesizing its input quantities using lumped parameter networks (LPNs) are described. In Section 4, typical numerical results are presented. Section 5 contains a summary and is followed by a list of references.

2. SOME COMMENTS ON SEM ANALYSIS OF CYLINDRICAL ANTENNA IN FREE SPACE

Integral equation formulations have proved to be a powerful tool in analyzing general shapes and impedance loading distributions on antennas and scatterers. Specifically for solving transient/broadband problems, a complex frequency ($s \equiv \Omega + j\omega$) approach employed by SEM analysis starting with a Hallén or Pocklington type of integral equation has been widely used. The basic idea of SEM is to express the electromagnetic behavior in terms of complex singularities. The SEM work reported to date has concentrated on computing the responses of finite size objects in free space. In many instances, only poles appear in the finite plane, giving rise to a considerable simplification. One example is the SEM analysis of a straight thin wire in free space. The first numerical study using the method of moment techniques, on the subject of SEM analysis of thin wires, was performed by Tesche [4.48].

In this reference [4.48], the exterior natural resonant frequencies were computed starting from the electric field integral equation formulation. The other SEM parameters, e.g., coupling coefficients and natural modes, were also evaluated at the natural frequencies. These SEM parameters are then used in determining the time domain behavior of the induced current on the thin wire scatterer. The interested reader is referred to Tesche's work [4.48] which lists a number of observations and questions, some of which are yet to be addressed. One important observation is that the transient behavior of this class of scatterers can be computed for any angle of incidence and shape of incident waveform by knowing the SEM parameters; natural frequencies, coupling coefficients and natural modes.

3. SEM ANALYSIS OF CYLINDRICAL ANTENNA IN CONDUCTIVE MEDIUM

3.1 Integral Equation Formulation

Consider a linear cylindrical dipole antenna of lengh $L = 2h$ and a radius, a, immersed in a homogenous, isotropic and time invariant conducting medium. The geometry of the problem is illustrated in Figure 1, where the constitutive parameters of the surrounding medium which is assumed non-magnetic are σ_1, $\varepsilon_0\varepsilon_r$, and μ_0. The Pocklington form of the integral equation for the axial current distribution is similar to that for the free space case, except for the change in the propagation constant and the additional conduction current component of the incident field. The integral equation applicable to the conducting medium case is now given by

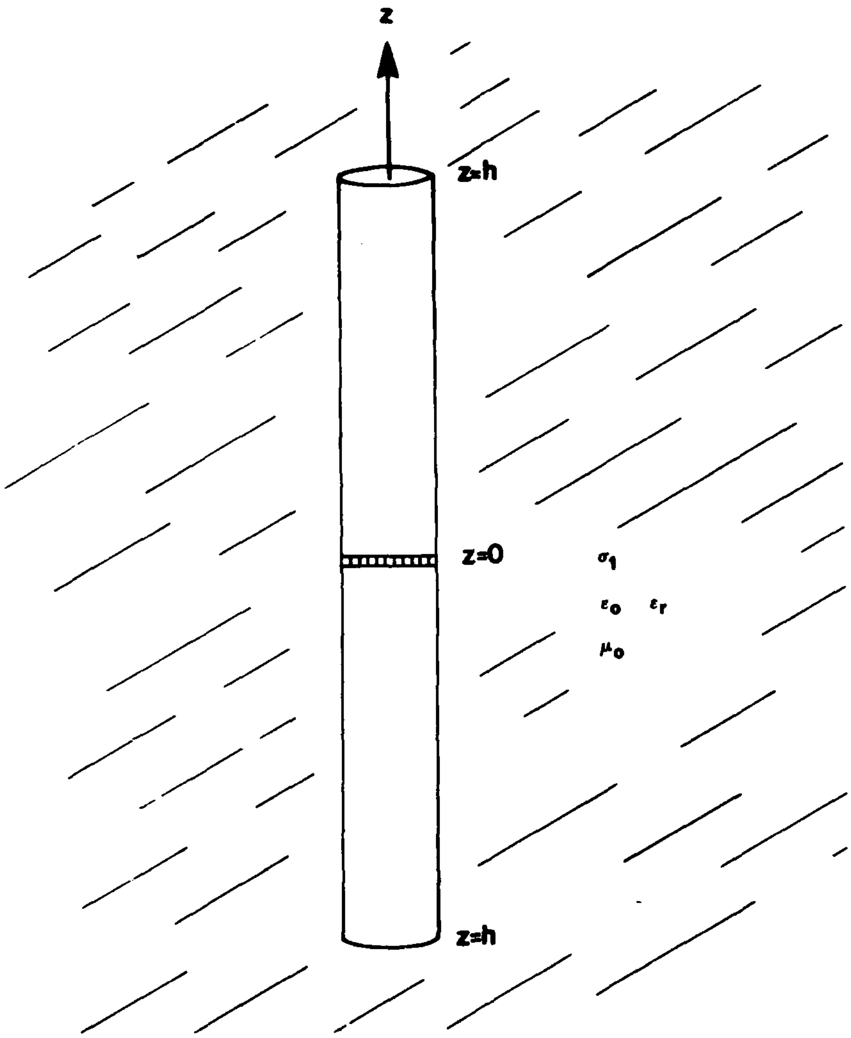


Figure 1. An isolated linear antenna in a constant conducting medium

$$\left(\frac{\partial^2}{\partial z^2} - \gamma_1^2\right) \int_{-h}^{h} I(z',s)\ K(z,z';s)\,dz' = -(s\varepsilon + \sigma_1)\,E_z^{inc}(z,s) \qquad (1)$$

where

$$K(z,z';s) = \frac{e^{-\gamma_1 R(z,z')}}{4\pi R(z,z')}$$

$$\gamma_1 \equiv \text{complex propagation constant} = \frac{s}{c}\sqrt{\varepsilon_r + \frac{\sigma_1}{s\varepsilon_0}} = \sqrt{s\mu_0}\,\sqrt{s\varepsilon + \sigma_1}$$

and $R(z,z') = [(z-z')^2 + a^2]$.

Starting with the EFIE above, the object is to perform a numerical s-plane SEM analysis, leading to a determination of the input quantities Y_{in} and I_{sc}, which are obtained directly from the solution I(z,s) of the integral equation.

This analysis may be carried out numerically by computing Y_{in} and I_{sc} in the complex s-plane via a moment method solution of the integral equation. Using this method for evaluating the complex antenna input functions and a pole/zero searching routine, the antenna response singularities in the finite complex s-plane are sought. The input functions are then expressed in terms of the known singularities as ratios of polynomials and, later, as a summation of partial fractions for synthesis. However, in marked contrast with the free space case, it is seen that due to the square root in the complex propagation constant, branch points occur in the s-plane, making the complex s-plane two sheeted for each branch point. It is possible to analytically determine the location of these branch points and then verify these locations numerically, using numerical methods. A detailed discussion of the occurrence of the branch points and the associated branch cut is the subject of the following subsection.

3.2 Occurrence of Branch Points

In the kernel function K(z,z',s) of the integral equation, the exponent $-\gamma_1 R$ is the source of branch points. Since the distance term R(z,z') is frequency independent, an examination of the propagation constant γ_1 suggests the occurrence of branch points at $s = 0$ and $s = -\sigma_1/(\varepsilon_0\varepsilon_r)$. For the case of real σ_1 and ε_r, the branch points occur in the normalized s-plane at $(sL/\pi c) = 0$ and $(sL/\pi c) \cong 120\,\sigma_1 L/\varepsilon_r$. The associated branch cut extends from the origin to the second branch point along the negative real axis in the s-plane. This was numerically verified for a number of cases by computing the discontinuity in the input impedance and admittance Z_{in} and Y_{in}, respectively, along the negative real axis. The discontinuities were seen to be purely imaginary and they did validate the predicted extent of the branch cut.

Of special interest, especially for synthesizing the antenna response using lumped networks, is the determination of the locations of poles and zeros, which fortunately turns out to be less tedius than one might expect. The procedure for obtaining the pole-zero locations for the lossy medium case, knowing their locations for the free space, is outlined below.

3.3 Trajectories of Poles and Zeros Given the Free Space Value

In this section, the relationship between the pole (or zero) locations for a linear antenna in free space and in a conducting medium is developed. This relationship is then used in plotting the pole and zero trajectories in the complex s-plane for the antenna input impedance, Z_{in}.

It has been observed [4.54] that the essential change in the integral equation for free space and for the conducting medium lies in the complex propagation constant. This change amounts to a change of variable from

s to $s\sqrt{\varepsilon_r + (\sigma_1/s\varepsilon_0)}$.

This observation leads to the following relationship

$$p_{\alpha 1} = \frac{-\sigma_1}{2\varepsilon_0\varepsilon_r} + \frac{1}{2\varepsilon_r}\sqrt{\left(\frac{\sigma_1}{\varepsilon_0}\right)^2 + 4\varepsilon_r p_{\alpha 0}^2} \tag{2}$$

where $p_{\alpha 1}$ and $p_{\alpha 0}$ are the location of poles of Z_{in} for the antenna in a lossy medium and free space, respectively.

To graphically illustrate this equation, consider a dipole antenna of length $L = 2h = 1\,\text{m}$ and shape factor $\Omega = [2\,\ell n(L/a)]$ of 10.59. Its first few poles and zeros are computed numerically for the free space situation. Note that the origin is an impedance pole when $\sigma = 0$. But as soon as some finite conductivity is introduced, the origin turns into a branch point and poles and zeros asymptotically move toward $-\infty$ on the negative real axis. The pole-zero trajectories are plotted in Figure 2, where the conductivity is gradually increased from 0 mhos/m to about 5×10^{-2} mhos/m. These trajectories are computed by using equation (2). With respect to these trajectories, two observations are in order, (i) the trajectories do not appear to intersect, so that there is no cancellation of a pole and zero for any finite value of conductivity, and (ii) the trajectories do not cross or run into the finitely long branch cut on the negative real axis.

In review, it is noted that once the pole-zero structure in the upper left half s-plane is known, it is straightforward to compute the corresponding pole-zero locations for the antenna input quantity, when the antenna is surrounded by a conducting medium. The determination of the singularity structure is complete, once the branch cut is added to the knowledge of pole-zero locations.

The occurrence of the finite branch cut along the negative real axis of the complex s-plane contributes to the pole series of antenna input quantity (say F(s)). F(s) can represent either Z_{in} or Y_{in} of the antenna in a conducting medium. Let us now determine the branch cut contribution to F(s). Consider the following contour integral, in reference to Figure 3

$$\int \frac{F(s)}{s_0 - s}\, ds = 2\pi j \left[F(s_0) + \sum_{\alpha} \frac{R_\alpha}{s_\alpha - s_0} \right]. \tag{3}$$

where R_α is the residue at pole s . The above result follows from Cauchy's residue theorem. Assuming negligible contribution on the infinite circle, the contour integral on the left side of the above equation becomes

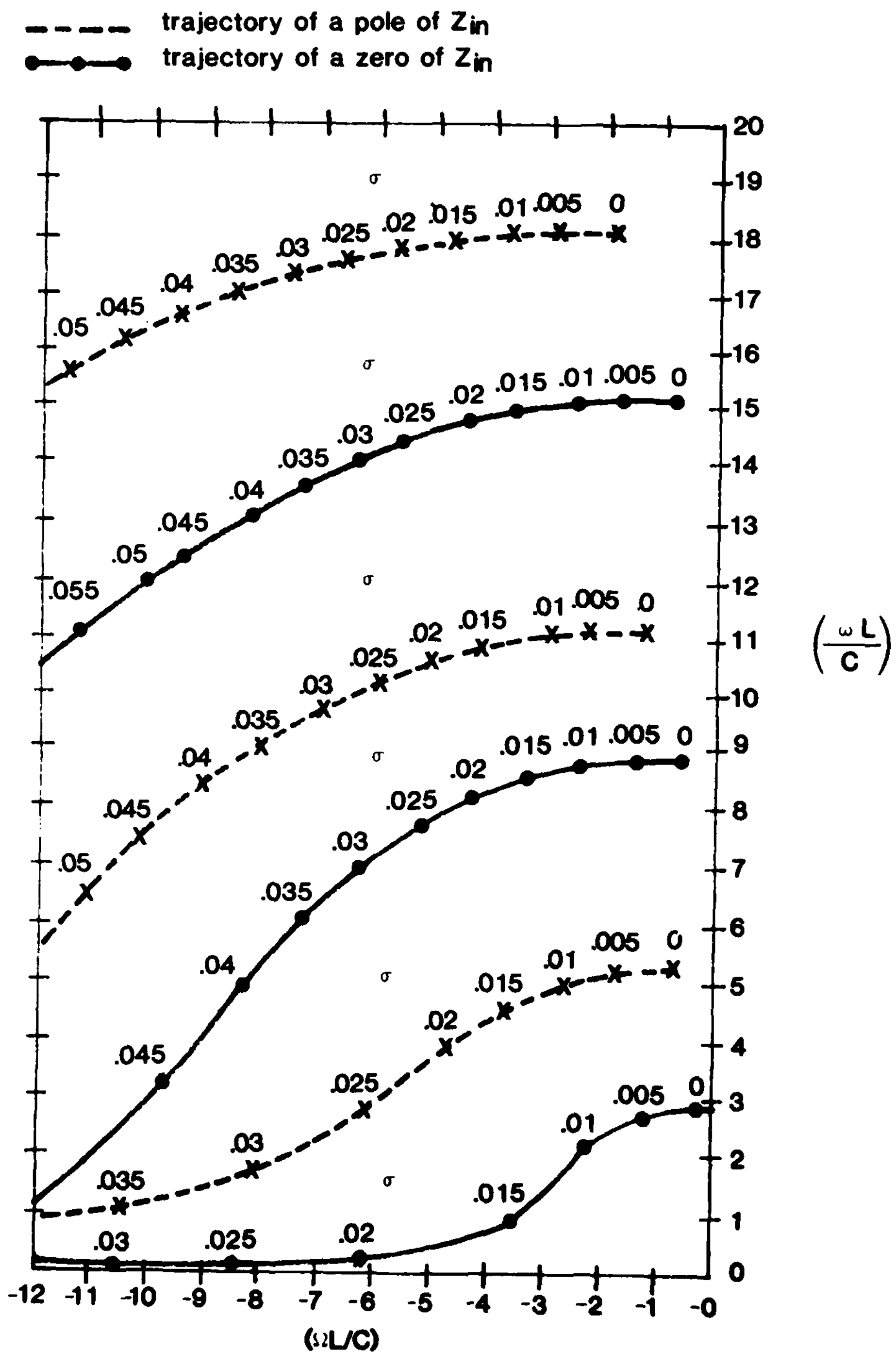


Figure 2. Pole and zero trajectories in the complex frequency of the input impedance Z_{in} of a linear antenna, for various medium conductivities

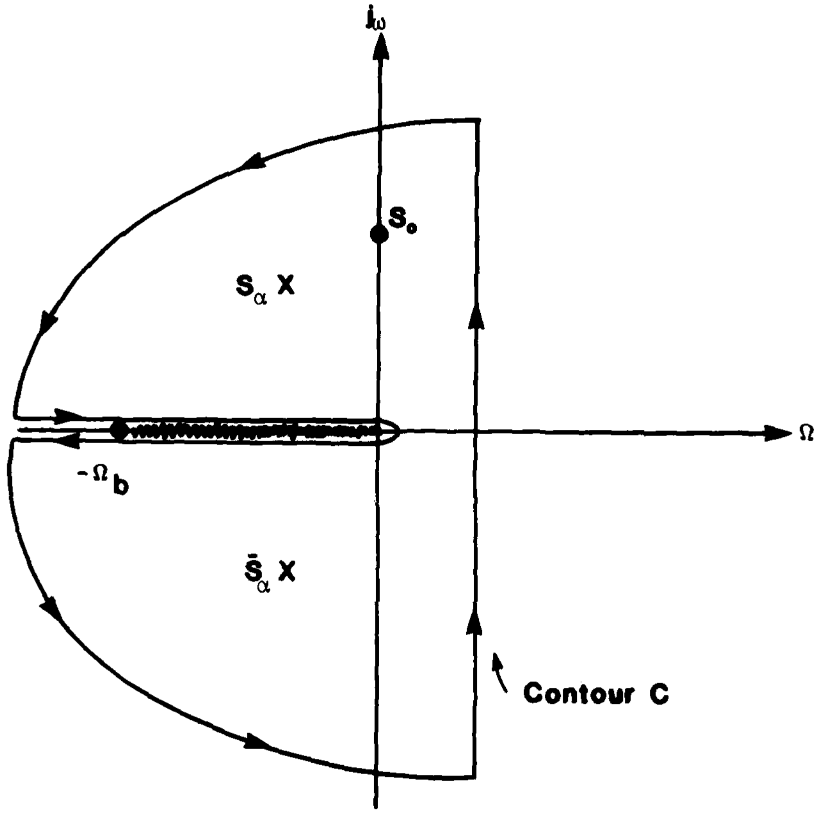


Figure 3. Evaluation of the branch cut contribution to the input quantities, e.g., Z_{in} or Y_{in}

$$\int_0^{-\Omega_b} \frac{\Delta F(\Omega)}{s_0 - \Omega}\, d\Omega = 2\pi j \left[F(s_0) + \sum_\alpha \frac{R_\alpha}{s_\alpha - s_0} \right] \tag{4}$$

leading to

$$F(s) = \sum_\alpha \frac{R_\alpha}{s - s_\alpha} + \frac{1}{2\pi j} \int_0^{-\Omega_b} \frac{\Delta F(\Omega)}{s - \Omega}\, d\Omega$$

where $\Delta F(\Omega)$ is the discontinuity across the branch cut. The integral in Eq. (4) is specifically the branch cut contribution to the response and the summation term is the familiar pole series contribution. It is interesting to compare the contribution to Y_{in} of the branch cut, relative to the collective contributions of the poles in the finite complex plane. Such a comparison was carried out for an antenna of 1 meter in length, and for all of the examples considered for $\sigma = 0$ to 10^{-2} mhos/meter, the branch cut contributions were found to be negligible.

3.4 I_{sc} and Y_{in} and Associated Network Realizations

The pole series of the form contained in Eq. (4) leads to partial fraction expansions of the following form the Y_{in} and I_{sc}.

$$Y_{in}(s) \cong \sum \frac{C_{y1}(\alpha)s^2 + C_{y2}(\alpha)s + C_{y3}(\alpha)}{s^2 + C_{i4}(\alpha)s + C_{i5}(\alpha)} \tag{5}$$

$$I_{sc}(s) \cong \sum \frac{C_{i1}(\alpha)s^2 + C_{i2}(\alpha)s + C_{i3}(\alpha)}{s^2 + C_{i4}(\alpha)s + C_{i5}(\alpha)} \tag{6}$$

The partial fraction expansions can be realized using lumped networks and the results are summarized below.

Defining

$$D_{y\alpha} = C_{y1}(\alpha) - [C_{y3}(\alpha)/C_{y5}(\alpha)];\ E_{y\alpha} = C_{y2}(\alpha) - [C_{y3}(\alpha)C_{y4}(\alpha)/C_{y5}(\alpha)]$$
$$F_{y\alpha} = C_{y4}(\alpha) - [D_{y\alpha}C_{y5}(\alpha)/E_{y\alpha}];\ G_{y\alpha} = E_{y\alpha} - D_{y\alpha}F_{y\alpha} \tag{7}$$

the LPN elements in Y realization of Figure 4a are given by

$$R_2 = C_{y5}(\alpha)/C_{y3}(\alpha);\ C_\alpha = E_{y\alpha}/C_{y5}(\alpha)$$
$$R_1 = 1/D_{y\alpha};\ L_\alpha = 1/G_{y\alpha} \text{ and } R_{0\alpha} = F_{y\alpha}/G_{y\alpha}\ . \tag{8}$$

Furthermore, the individual current sources I_1, I_2, I_3, ... I_m of I_{sc} in

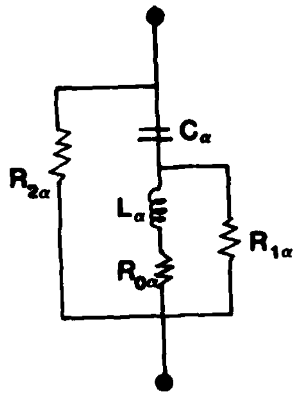


Figure 4a. Y_α corresponding to a pole pair

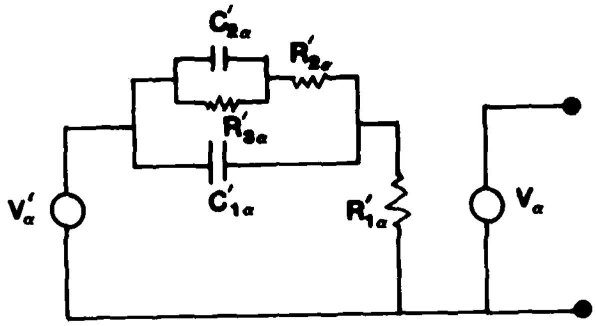


Figure 4b. V_α corresponding to a pole pair

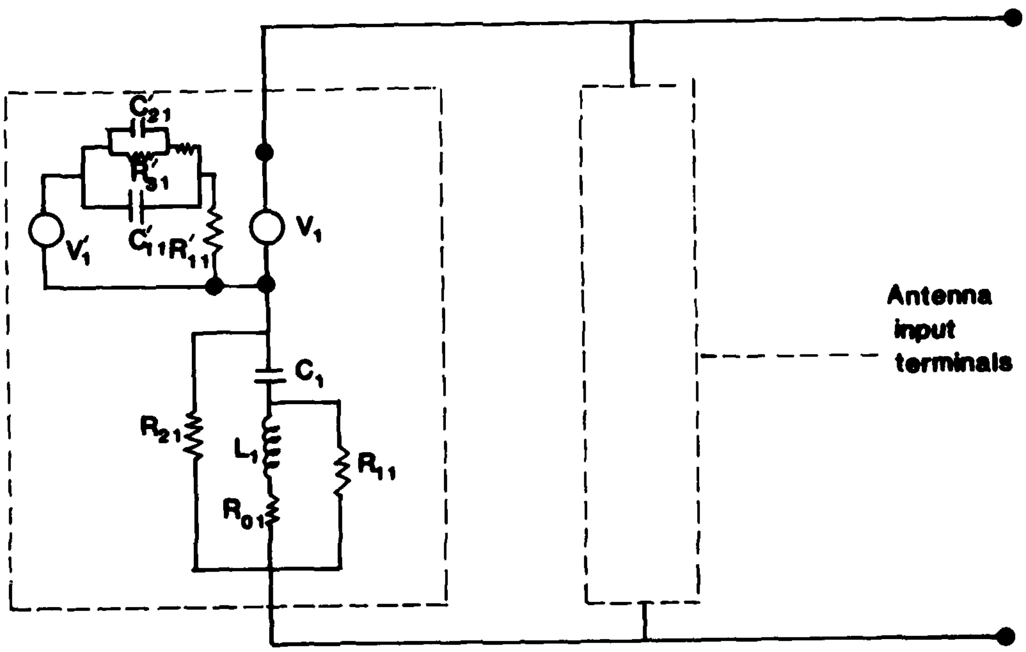


Figure 4c. Realized circuit using Norton approach

Eq. (6) are combined with individual Y_1 Y_2, Y_3, ... Y_M in Eq. (5), by writing the equivalent voltage sources V_1, V_2, V_3, ... V_M as indicated.

$$V_\alpha = \frac{I_\alpha}{Y_\alpha} = \frac{s^2\, C_{i1}(\alpha) + s\, C_{i2}(\alpha) + C_{i3}(\alpha)}{s^2\, C_{y1}(\alpha) + s\, C_{y2}(\alpha) + C_{y3}(\alpha)} \tag{9}$$

The V_α are synthesized in a form given by Figure 4b, denoting

$$\begin{aligned} &A_{v\alpha} = C_{i1}(\alpha)/C_{i2}(\alpha); \qquad B_{v\alpha} = C_{i3}(\alpha)/C_{y2}(\alpha) \\ &D_{v\alpha} = C_{i2}(\alpha) - [C_{i1}(\alpha)/A_{v\alpha}]; \qquad E_{v\alpha} = C_{i3}(\alpha) - [C_{i1}(\alpha)B_{v\alpha}/A_{v\alpha}] \\ &F_{v\alpha} = 1 - [A_{v\alpha}E_{v\alpha}/D_{v\alpha}]; \qquad G_{v\alpha} = E_{v\alpha} - [D_{v\alpha}B_{v\alpha}/F_{v\alpha}] \; . \end{aligned} \tag{10}$$

In terms of the above known parameters, the LPN elements in the source synthesis are given by

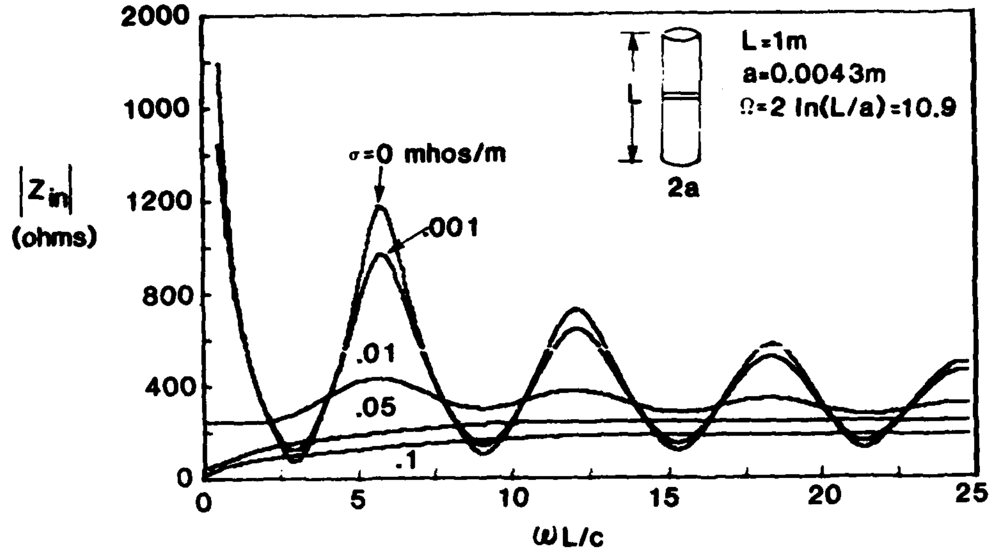
$$\begin{aligned} &V_\alpha = C_{i1}(\alpha)/C_{y1}(\alpha); \qquad R_{1\alpha} = C_{i1}(\alpha)/A_{v\alpha}; \qquad R_{2\alpha} = D_{v\alpha}/F_{v\alpha} \\ &R_{3\alpha} = G_{v\alpha}/B_{v\alpha}; \qquad C_{1\alpha} = A_{v\alpha}/D_{v\alpha}; \qquad F_{v\alpha}/G_{v\alpha} \; . \end{aligned} \tag{11}$$

Thus, the complete Norton circuit realization of the antenna in a lossy medium is shown in Figure 4 and all of the LPN elements as well as sources are known, in terms of pole locations and residues. It is observed that this is only one form of the circuit realization, since synthesis in general is a non-unique process. It is also noted that if σ_1 of the ambient medium were = 0, the circuit realization would simplify considerably, giving equivalent circuits that are similar in form to those used by other investigators [4.2].

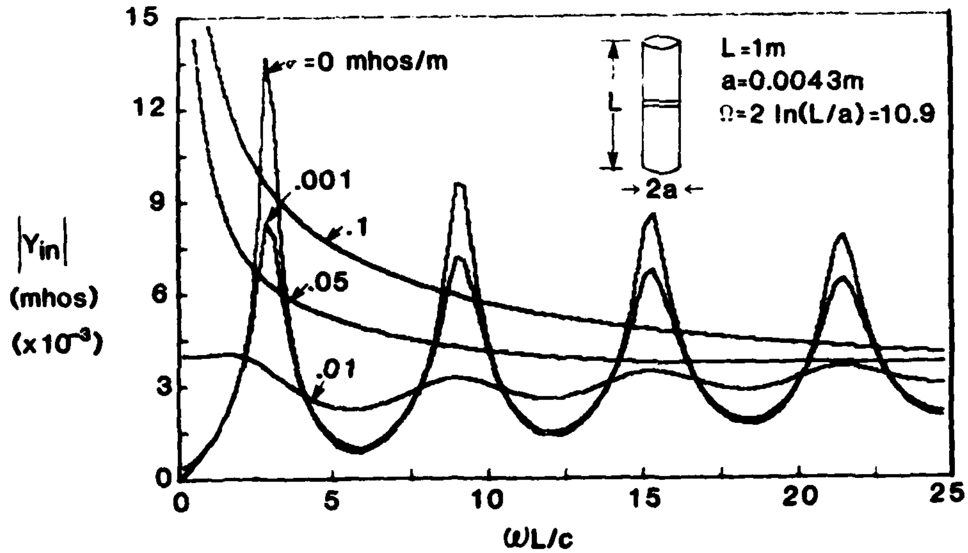
4. NUMERICAL RESULTS

4.1 Input Impedance and Admittance

Using the previously described method of analysis, a series of calculations were performed for an isolated antenna (L = 1 meter, Δ/L = 0.05, Ω = 10.59) in a conducting region. Figure 5a shows the magnitude of the antenna input impedance as a function of frequency ($\omega L/c$) for conductivities of σ = 0, .001, .01, .05, and .1 mhos/meter. The corresponding input admittance magnitude is shown in Figure 5b. As the conductivity increases from zero, it is seen that the input impedance starts having a resistive component, and at a high conductivity value, the antenna seems to be "shorted out." It is interesting to note that the resonance effects of the antenna vanish as σ increases. At σ = .01 mhos/meter, the resonances have almost been eradicated, and at σ = 0.5 mhos/meter, there is no trace of a resonant behavior. It is apparent from Figure 2 what is happening in this case. As σ increases, the poles and zeros move off into the left hand complex frequency pland and give smaller contributions to the overall response. As seen in Figure 2, the change of pole and zero locations from σ varying between .01 and .05 mhos/meter is extreme, especially for the fundamental (lowest) resonance.



(a)



(b)

Figure 5. Plots of the input impedance magnitude (a) and the input admittance magnitude (b) for isolated linear antenna in a conducting medium

For the driven antenna problem, we are interested in more than just the input impedance or admittance. We must also calculate the Thevenin or Norton equivalent sources. For this example, Figure 6 shows the open circuit voltage magnitude, V_{oc}, and short circuit current magnitude, I_{sc}, as a function of frequency ($\omega L/c$) for $\sigma = 0.001$ mhos/meter and various angles of incidence θ (defined to be 0^o along the antenna, and 90^o normal to it) of a 1 volt/meter incident plane wave. It is interesting to note that at $\omega = 0$, the short circuit current of the antenna goes to zero in the case of no air conductivity. As σ is increased, however, the current is non-zero. In a manner similar to Y_{in} and Z_{in}, the resonance effects in I_{sc} and V_{oc} tend to be swamped out as σ increases.

4.2 Synthesis Results for an Antenna in Free Space

As previously discussed, the necessary information to perform a lumped circuit synthesis of the antenna is contained in the poles and residues of Y_{in} and I_{sc}. In carrying out the reconstruction of Y_{in} or I_{sc} from the appropriate poles and residues of the functions, it is important to keep in mind that we are attempting to approximate a function having an infinite number of poles by one with a finite number. It has been found that for a pole series representation for Y_{in} of the form

$$Y_{in} = \sum_{\substack{\alpha \text{ pole} \\ \text{pairs}}} \left(\frac{R_\alpha}{s - s_\alpha} \right) \tag{12}$$

the convergence of the series to the correct value is a very slow function of the number of pole pairs considered.

One approach to alleviate this problem is to use a modified pole series expansion [4.3] of the form

$$Y_{in} \approx \sum_{\substack{\alpha \text{ pole} \\ \text{pairs}}} \left(\frac{R_\alpha}{s - s_\alpha} + \frac{R_\alpha}{s_\alpha} \right) \tag{13}$$

This function has the same poles and residues as Y_{in}, but is also constrained to vanish term by term at $\omega = 0$. It has been verified that the resulting spectrum magnitude for $|Y_{in}|$ as computed from the integral equation and as computed from the modified pole series are in excellent agreement. As a result, the synthesis of the circuits for Y_{in}, as well as for the other input quantities, is based on the modified pole expansion form.

Using the synthesis procedure previously outlined, the following circuit elements for the input admittance of the isolated antenna in free space ($L = 1$ meter, $\Delta/L = .05$, $\Omega = 10.59$, $\sigma = 0$) have been determined (see Table 1). Similarly, the elements for the Norton sources are given in Table 2. For the form of the circuits, the reader is referred to Figure 4.

As may be noted, there are several negative circuit elements which occur in LPN. This is the unfortunate consequence of attempting to model a distributed field problem by a discrete circuit model. Efforts by other investigators [4.46] have led to a circuit synthesis involving only positive definite elements, but the circuits are relatively complicated.

It is interesting to consider the effects of neglecting the negative resistance in the synthesized circuits for Y_{in}. We found that neglecting the negative elements introduces a significant error for the higher resonances,

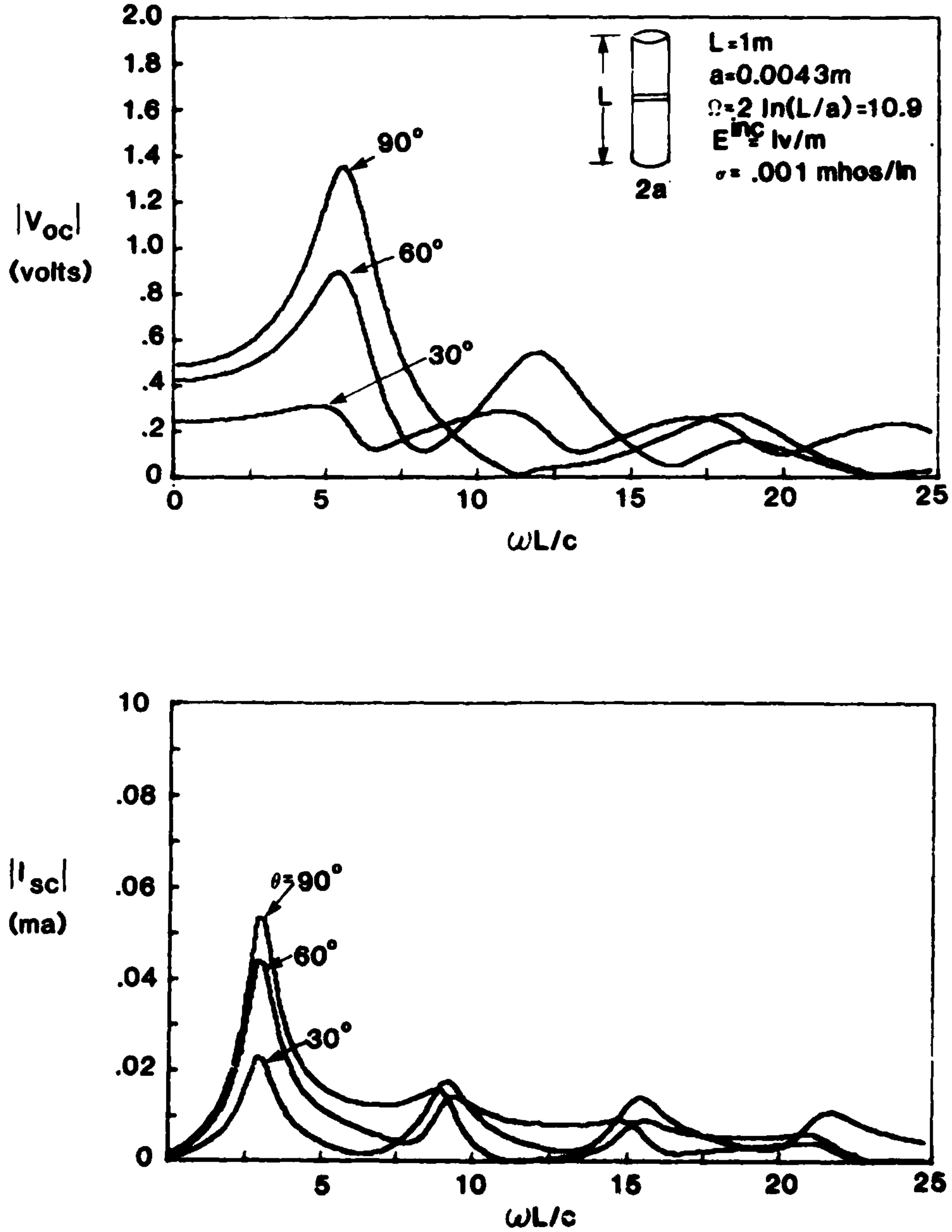


Figure 6. Plots of magnitudes of (a) open circuit voltage and (b) short circuit current for σ = 0.001 mhos/meter and various angles of incidence

but for low frequencies (i.e., the first natural frequency and below) the effects are not severe. Also, the results of LPN model of Y_{in} and Z_{in} were compared with the 'true' solution of the integral equation revealing discrepancies in higher frequencies due to truncation of pole series.

TABLE 1

NORTON CIRCUIT ELEMENTS FOR ISOLATED ANTENNA IN FREE SPACE

Pole Pair	$R_0(\Omega)$	$R_1(k\Omega)$	$R_3(k\Omega)$	$L(\mu\ )$	C(pf)
1	-3.05	2.31	∞	.47	2.77
2	-98.28	6.03	∞	.41	.33
3	-217.30	9.43	∞	.39	.12

TABLE 2

NORTON CIRCUIT ELEMENTS FOR SOURCES FOR ISOLATED ANTENNA IN FREE SPACE ($\theta^{inc} = 90°$)

Pole Pair	V'(Volts)	R'_1(Ohms)	R'_2(Ohms)	R'_3(Ohms)	$C'_1(\mu f)$	$C'_2(\mu f)$
1	.64	7.32×10^{-3}	2.97×10^{-4}	0	.66	∞
2	-.31	5.75×10^{-3}	2.29×10^{-3}	0	.073	∞
3	.23	5.15×10^{-3}	3.14×10^{-3}	0	.022	∞

4.3 Synthesis for an Antenna in a Lossy Medium

The various circuits used for representing the impedance or admittance as well as the sources for an antenna in a lossy medium have been discussed in Section 3. It had been postulated that the presence of a finite air conductivity would give rise to a simple resistive element shunting all capacitors in the LPNs for the antenna. The rationale behind this speculation stems from the fact that if a lossy dielectric is inserted into an ideal capacitor, the resulting circuit model for the device is a shunt R-C circuit with $R = [\sigma/(C\varepsilon)]$.

We investigated the validity of this alternate circuit representation by calculating the shift of the natural frequency of the LPN for Y_{in} given in Figure 4a with that given by simplified circuit involving only shunt resistances. In this case, both circuits are identical for $\sigma = 0$. As σ increases, all elements of the circuit in Figure 4a change, but in the approximate circuit, only the shunt resistance varies. The resulting difference between the actual antenna resonance frequency and that provided by the simplified LPN is significant, especially at conductivities greater than 0.01 mhos/meter. There are also difficulties with this approach of using only shunt resistances in attempting to synthesize the antenna response. The implication of the above is that all LPN elements must vary as σ varies.

As an example of a specific synthesis for an antenna in a conductive region, consider an isolated antenna ($L = 1$ meter, $\Delta/L = .05$, $\Omega = 10.95$) in a region with $\sigma = .001$ mhos/meter. The following table presents the circuit elements for representing Y_{in} of Figure 4.

TABLE 3

NORTON CIRCUIT ELEMENTS FOR ISOLATED ANTENNA ($\sigma = 10^{-3}$ mhos/meter)

Pole Pair	$R_0(\Omega)$	$R_1(k\Omega)$	$R_2(k\Omega)$	$L(\mu H)$	$C(pf)$
1	105.9	8.35	3.19	.48	2.73
2	-5.4	7.74	27.06	.41	.33
3	-128.0	10.81	74.15	.40	.12

Similarly, the source elements for each pole pair for $\theta = 90°$ excitation are given by Table 4.

TABLE 4

CIRCUIT ELEMENTS FOR SOURCES FOR ISOLATED ANTENNA ($\sigma = 10^{-3}$ mhos/meter), $= 90°$

Pole Pair	V'(volts)	$R'_1(\Omega)$	$R'_2(\Omega)$	$R'_3(\Omega)$	$C'_1(\mu f)$	$C'_2(\mu f)$
1	.64	7.56×10^{-3}	3.05×10^{-3}	0	.64	∞
2	-.31	5.84×10^{-3}	2.31×10^{-3}	0	.041	∞
3	.23	5.21×10^{-3}	3.16×10^{-3}	0	.022	∞

As may be noted, the values of R'_3 and C'_2 are such that these circuit elements appear shorted in the source circuits for this value of conductivity.

Figure 7 presents a comparison of $|Y_{in}|$ as computed from the integral equation and as obtained from the LPN. As before, good agreement is noted for low frequencies, but the high frequency errors are again present. These errors arise from not having a sufficient number of LPN circuits and, in part, due to neglecting the branch cut contribution to the admittance, although for this value of conductivity, this contribution is small.

5. SUMMARY

In this paper, we have presented an SEM analysis to understand the behavior of a linear antenna in a conducting region. In the past, theoretical studies of SEM have been applied to relatively simple metallic objects, with only pole type of singularities. For the present problem, branch points and an associated branch cut are present in the complex frequency plane. The occurrence of branch points and the finite extent of the branch cut are numerically validated. However, for the examples of conductivities considered, the contribution of the branch cut to input parameters (Y_{in} or Z_{in}), compared with the combined contribution of complex poles in the finite s-plane was found to be negligible. This is not always the case and, in general, branch cut contributions should be included.

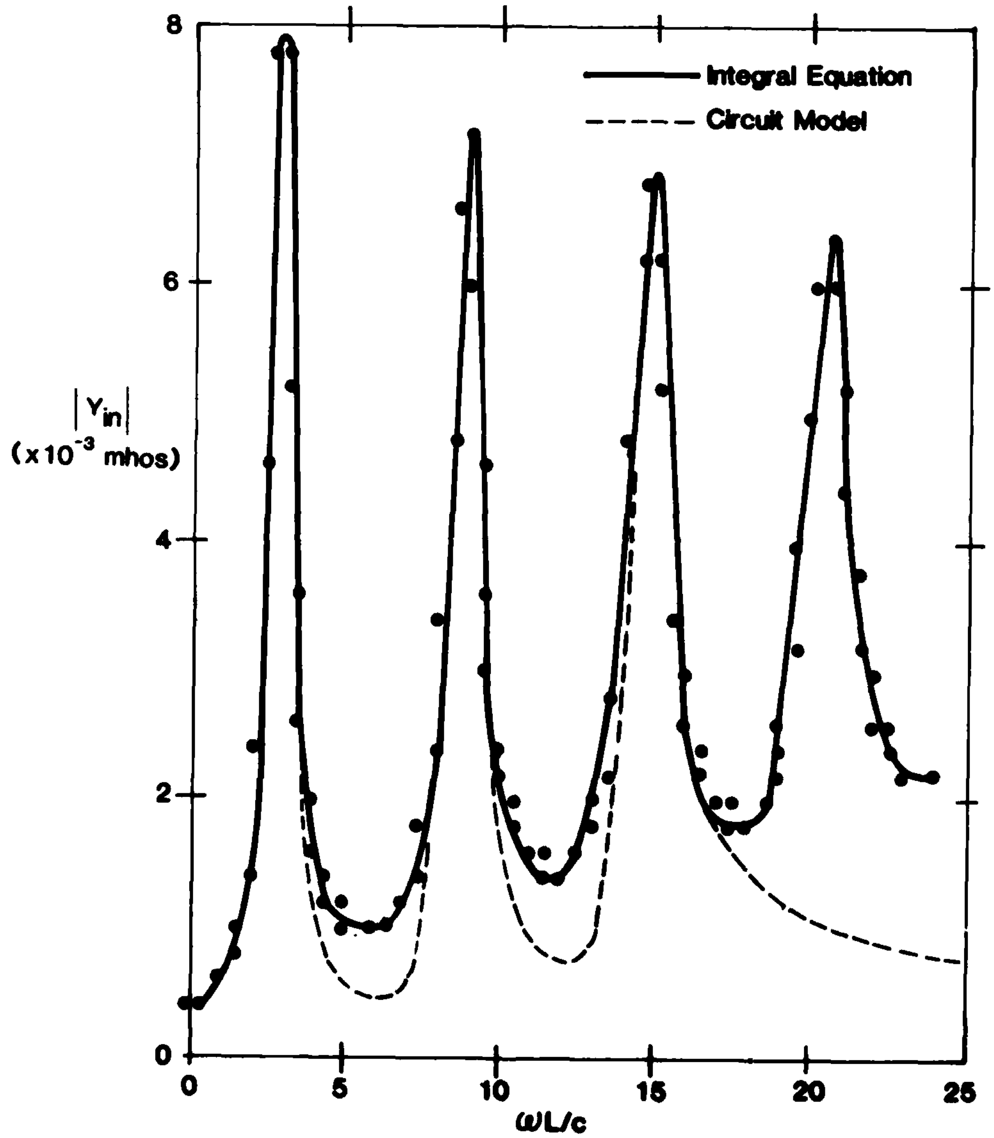


Figure 7. Plot of $|Y_{in}|$ vs. frequency for antenna in lossy medium ($\sigma = 10^{-3}$ mhos/meter)

Of specific interest in this paper is the determination of Y_{in} and I_{sc} of antennas in free space and in conducting media and their eventual realization into a Norton equivalent circuit for the antenna input terminals. Such circuits have their usefulness in the EMP interaction problem, for example. It was found that in the particular form of synthesis, non-negligible negative elements appear. If such circuit realizations are unacceptable, more complicated networks are possible. Furthermore, one can only synthesize pole type singularities and, hence, branch cut contributions to input quantities, if any, can be approximated by a finite number of poles for synthesis purposes. Another approach that deserves deeper investigation is to synthesize branch cut contributions by lossy transmission lines resulting in hybrid (lumped and distributed) parameter networks (HPNs). The applicable range of validity of neglecting the branch cut contributions deserves attention in future analytical efforts.

In conclusion, the past work has been extended for antennas in lossy media for the case of oblique incidence. Certain symmetries are preserved in the formulation of the problem, construction of the pole series, and synthesis. This results in circuit realizations that are consistent with each other. For example, if the medium conductivity is removed, the circuit realizations for all input quantities are consistent with the free space problem.

RESONANCES AND SURFACE WAVES: THE INVERSE SCATTERING PROBLEM

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The Singularity Expansion Method (SEM) of radar scattering[1] [2.1, 3.1] is based on the observation that the echo return from pulsed radar signals consists of a superposition of damped sinusoids. In the frequency domain, the scattering amplitude consequently contains a number of complex poles, whose residues determine the amplitudes of the damped sinusoids. Along the real axis of physical frequencies, these poles manifest themselves as finite resonances in the echo, like the foothills of more or less distant, very high mountain peaks[2]. Both these descriptions of the scattering process, in the time domain or in the frequency domain, are equivalent, but each has its own advantages for extracting the wealth of information contained in the radar echoes as regards the target identification problem. In addition, we shall here introduce also the mode number domain.

In the time domain, the sinusoidal echoes are preceded by a pulse which is a replica of the incident pulse, being due to specular reflection. In the frequency domain, the specular echo appears as a non-resonant background, interfering with the resonant terms[3-5] [4.6, 4.12, 4.16, 4.20, 4.22, 4.35, 4.36, 4.37, 4.42,

1. Reference numbers refer to the collective bibliography in this issue.
2. Langenberg, K. J., "Methods and Applications in Transient Analysis", in Proceedings of the <u>International U.R.S.I. Symposium 1980 on Electromagnetic Waves</u>, Munich, Germany, August 26-29, 1980, p. 413 A/1.
3. Gaunaurd, G. C., and H. Überall, "Theory of Resonant Scattering from Spherical Cavities in Elastic and Viscoelastic Media", <u>J. Acoust. Soc. Amer</u>., Vol. 63, p. 1699, 1978.
4. Gaunaurd, G. C., and H. Überall, "Numerical Evaluation of Modal Resonances in the Echoes of Compressional Waves Scattered from Fluid-Filled Spherical Cavities in Solids", <u>J. Appl. Phys</u>., Vol. 50, p. 4642, 1979.
5. Flax, L., and H. Überall, "Resonant Scattering of Elastic Waves from Spherical Solid Inclusions", <u>J. Acoust. Soc. Amer</u>., Vol. 67, p. 1432, 1980.

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6.60]. This will be demonstrated here for the example of a conducting sphere of radius b, coated with a homogeneous dielectric of outer radius a.

A plane wave $\propto E_o \exp(ik_o z)$, incident on a spherical target, gives rise to a scattered far field[6] (using spherical coordinates r, θ, ϕ):

$$\vec{E}_{sc} = E_o (e^{ik_o r}/k_o r)\left[\hat{e}_\theta S_1(\theta)\cos\phi - \hat{e}_\phi S_2(\theta)\sin\phi\right], \tag{1}$$

where the polarization functions

$$S_1(\theta) = -i\sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{n(n+1)}\{a_n \frac{P_n^1(\cos\theta)}{\sin\theta} - b_n \frac{dP_n^1(\cos\theta)}{d\theta}\}, \tag{2a}$$

$$S_2(\theta) = -i\sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{n(n+1)}\{a_n \frac{dP_n^1(\cos\theta)}{d\theta} - b_n \frac{P_n^1(\cos\theta)}{\sin\theta}\} \tag{2b}$$

contain the "Mie coefficients" ($x=k_o a$):

$$a_n = -\frac{xj_n(x) - iZ_n[xj_n(x)]'}{xh_n^{(1)}(x) - iZ_n[xh_n^{(1)}(x)]'}, \tag{3a}$$

$$b_n = -\frac{xj_n(x) - iY_n[xj_n(x)]'}{xh_n^{(1)}(x) - iY_n[xh_n^{(1)}(x)]'}, \tag{3b}$$

with normalized impedances Z_n and admittances Y_n appropriate for the coated conducting sphere[6]. The complex-frequency poles are the roots of the denominators of a_n (TE modes) and b_n (TM modes) in the x variable ("characteristic equations"), to be designated $x_{n\ell}^{TE}$ and $x_{n\ell}^{TM}$, respectively. Here, n labels the mode and ℓ the multiplicity of solutions within a given mode.

Figure 1 shows the radar cross section

$$\sigma = (4\pi/k_o^2)\left|\sum_{n=1}^{\infty} (-1)^n (n+\tfrac{1}{2})(a_n - b_n)\right|^2 \tag{4}$$

plotted vs. x for a coating with $\varepsilon=6$ and relative thickness $\delta \equiv (a-b)/a = 0.05$, with clearly visible resonances. (This figure is very close to one previously obtained by Rheinstein [4.41].) The real parts of the pole positions, obtained by solving the characteristic equations, are indicated by arrows, labeled by $nM\ell$ (TM modes appearing in this x-region only).

It is evident that the resonances appearing in Fig. 1 interfere with some non-resonant background. Mathematically, a_n and b_n can be split accordingly, e.g.

$$b_n = \tfrac{1}{2}\exp(2i\xi_n^{TM})\{\frac{z_n^{(1)} - z_n^{(2)}}{iY_n - z_n^{(1)}} + 2i\exp(-i\xi_n^{TM})\sin\xi_n^{TM}\}, \tag{5a}$$

6. Ruck, G. T., et al., Radar Cross Section Handbook, Plenum, NY, 1970.

where

$$z_n^{(i)} = xh_n^{(i)}(x)/\left[xh_n^{(i)}(x)\right]', \quad i=1,2, \tag{5b}$$

and ξ_n^{TM} is defined by

$$S_n^{(o)TM} \equiv e^{2i\xi_n^{TM}} = -\left[xh_n^{(2)}(x)\right]'/\left[xh_n^{(1)}(x)\right]', \tag{5c}$$

this quantity representing the "S function" for a conducting sphere of radius a (the S function for a general layered sphere being defined by $S_n^{TE} = 1+2a_n$, $S_n^{TM} = 1+2b_n$). The characteristic TM equation is $iY_n = z_n^{(1)}$ so that in Eq. (5a), the first term in brackets is a meromorphic function representing the resonances. The second term gives a contribution to b_n corresponding to the case of a conducting sphere of radius a (where $Z_n \to 0$, $Y_n \to \infty$), and constituting a background to the resonant first term.

We have shown recently [6.60] that the S function of the layered sphere may be expressed in terms of a meromorphic function

$$R_n^{TM}(x) = \sum_{\ell=-\infty}^{\infty} \frac{(\gamma_{n\ell}^{TM})^2}{x-\hat{x}_{n\ell}^{TM}} \tag{6}$$

known in Nuclear Physics as "Wigner's R function" [6.51], thus determining its singularity structure. For the case of well-separated resonances one may use the "one-level approximation", which transforms Eq. (5a) into[7]

$$b_n = b_n^{(int)} + b_n^{(o)}, \tag{7a}$$

$$b_n^{(int)} = \exp(2i\xi^{TM})\, i\sum_{\ell} \frac{-\frac{1}{2}\Gamma_{n\ell}^{TM}}{x-x_{n\ell}^{TM} + \frac{1}{2}i\Gamma_{n\ell}^{TM}}, \tag{7b}$$

the resonance positions $x_{n\ell}^{TM}$ and widths $\Gamma_{n\ell}^{TM}$ being given explicitly [6.60] in terms of $\gamma_{n\ell}^{TM}$ and $\hat{x}_{n\ell}^{TM}$ of Eq. (6), and where

$$b_n^{(o)} = i\exp(i\xi_n^{TM})\sin\xi_n^{TM}. \tag{7c}$$

In Eqs. (7), $b_n^{(int)}$ represents a contribution of "internal" resonances which would be absent for a conducting sphere. Physically, the origin of this series of resonances lies in the excitation, during the scattering process, of a set of "internal" surface waves which propagate around the sphere inside the dielectric coating, and which at a given resonance frequency have phases that match up after each repeated circumnavigation, hence

7. Gaunaurd, G. C., and H. Überall, "R-Matrix Theory of Sound Scattering from Fluid Spheres via the Mittag-Leffler Expansion", J. Acoust. Soc. Amer., Vol. 68, p. 1850, 1980.

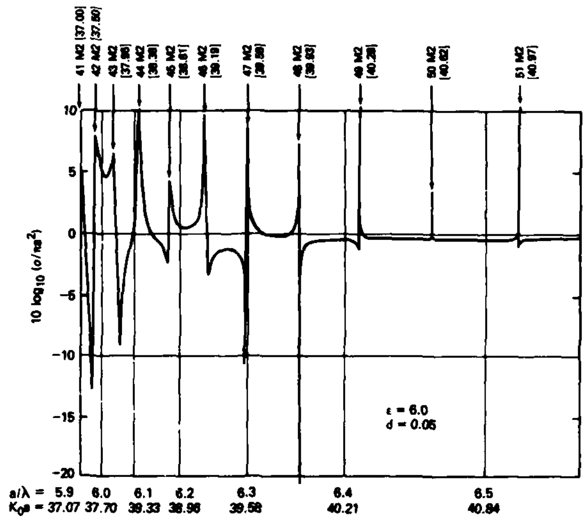

Fig. 1. Radar cross section for a conducting sphere coated with a dielectric (ε=6) of relative thickness δ=0.05, plotted vs. $x \equiv k_o a \equiv 2\pi a/\lambda$. Resonances are labeled by $nM\ell$ (n=mode number, M="transverse magnetic", ℓ= resonance order) and value of $k_o a$.

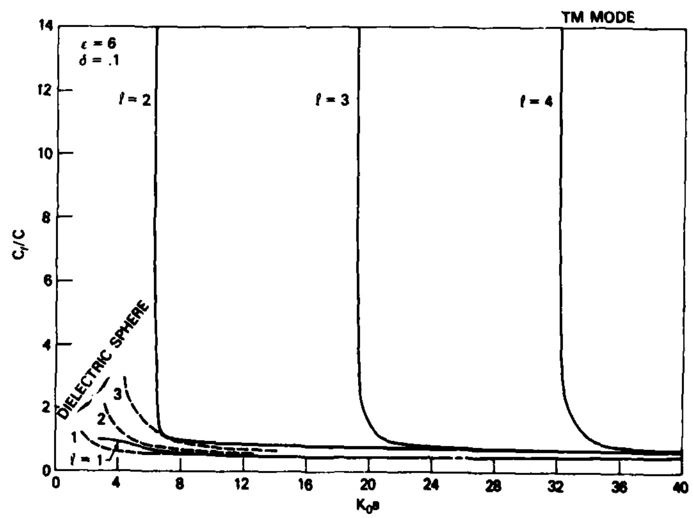

Fig. 2. Dispersion curves $c_\ell(x)/c$ for the TM-mode surface waves no.ℓ=1,2, and 3 on a conducting sphere with dielectric coating, of outer radius a. For comaprison, dispersion curves of TM surface waves on a dielectric sphere of radius a are shown also.

building up the resonance. The existence of the surface waves is demonstrated by transforming the SEM (frequency) poles of Eq. (7b) into poles in the complex mode number (n) plane, located at[4] [6.60, 6.61, 6.104]

$$\hat{n}_\ell = n_\ell + \tfrac{1}{2} i \hat{\Gamma}_{n\ell} \qquad (8a)$$

where n_ℓ is defined by the equation $x_{n_\ell \ell} = x$, the resonance frequencies $x_{n\ell}$ being considered a function of n (see fig. 1), and where $\hat{\Gamma}_{n_\ell \ell} = \Gamma_{n_\ell \ell}/(dx_{n_\ell \ell}/dn_\ell)$.

Evaluating Eqs.(2) at these poles shows their θ dependence to be $\exp \pm i (\hat{n}_\ell+\tfrac{1}{2})\theta$, so that they represent surface waves with phase velocities

$$c_\ell(x) = \left[x/(n_\ell + \tfrac{1}{2})\right] c. \qquad (8b)$$

The corresponding dispersion curves for the surface waves can then be obtained from the frequency resonances, and are shown in Fig.2 for the TM type surface waves. The families of resonances $x_{n\ell}$ (for the given ℓth surface wave) recurring in successive modes n are the physical manifestations of the surface waves, the latter causing the resonances by phase-matching after successive circumnavigations as seen from Eq. (8b), and taking into account a π/a phase jump at each of the two convergence points of the surface waves on the sphere.

We next interpret the term $b_n^{(o)}$ of Eqs. (7) which at first sight appears to be a non-resonant background term, possibly identical with the entire function often postulated [3.1] in SEM to contribute to the scattering amplitude in addition to the resonant terms $b_n^{(int)}$. In reality, however, this term is also resonant since it reads

$$b_n^{(o)} = -\tfrac{1}{2} \frac{[xj_n(x)]'}{[xh_n^{(1)}(x)]'} . \qquad (8c)$$

Its poles are the well-known [3.1] complex zeros $x_{n\ell}^{(o)TM}$ of $[xh_n^{(1)}(x)]'$, and expanding the latter expression about these zeros, the TM contribution from $b_n^{(o)}$ e.g. to $S_1(\theta)$ of Eq. (2a) may be written as

$$S_1^{(o)TM}(\theta) = -ix\hat{\varepsilon}_\theta \cos\phi \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{n(n+1)} \frac{1}{\hat{x}_{n\ell}^{(o)TM}} \frac{[\hat{x}_{n\ell}^{(o)} j_n(\hat{x}_{n\ell}^{(o)})]'}{[\hat{x}_{n\ell}^{(o)} h_n^{(1)}(\hat{x}_{n\ell}^{(o)})]^{\bullet}} \qquad (8d)$$

$$\times \frac{dP_n^1(\cos\theta)}{d\theta} \frac{1}{x-\hat{x}_{n\ell}^{(o)TM}} ,$$

which exhibits (via the last factor) the meromorphic character of even this "background" contribution. The corresponding resonances, however, are generally very broad due to the large imaginary parts [3.1] of $x_{n\ell}^{(o)TM}$, so that in a plot of cross section vs. frequency, they would not appear as resonances, Nevertheless, no entire function seems to be present in this case.

Analogously to the "internal" surface waves, the resonances in $b_n^{(o)}$ likewise have an interpretation in terms of diffracted, damped "external surface waves", also called, "creeping waves" by Franz[8] who obtained them explicitly by applying the Watson transformation to the scattering amplitude of the conducting sphere. He showed that in addition, a specularly reflected contribution

$$S_1^{(o)\,spec}(\theta) = -\tfrac{1}{2}xe^{-2ix\cos\theta/2} \tag{8e}$$

got added to the creeping-wave contribution in $S_1^{(o)}$ of Eq. (8d), which arose from a saddle-point contribution to the integral into which the mode sum of Eq. (2) was converted by the Watson transformation.

In the present case of the coated sphere, there is another "geometrical" contribution present in addition to the reflected wave, namely refracted (or transmitted) waves that penetrate the interior of the coating from which they re-emerge; these have been studied for the acoustic case[9] previously. Summarizing, therefore, we see that in the scattering process, resonances are generated both by internal and to a lesser degree by external (diffracted) surface waves, while "geometrical" specularly reflected as well as transmitted (refracted) waves produce an additional, possibly resonant contribution to the scattering amplitude.

The locations $x_{n\ell}$ and widths $\Gamma_{n\ell}$ of the frequency resonances have been shown to provide an analytic solution to the inverse scattering problem for the example considered [4.15, 4.17, 4.18], i.e. to determine the thickness and the dielectric constant ε of the coating on the conducting sphere. Using asymptotic forms of the Bessel functions, one finds e.g.

$$\varepsilon^{\frac{1}{2}} = |\cot(\pi X_{n\ell}^{TM} / \Delta_{n\ell}^{TM})| \tag{9a}$$

and

$$\delta/a = (\pi/\Delta_{n\ell}^{TM})\,|\tan(\pi X_{n\ell}^{TM}/\Delta_{n\ell}^{TM}) \tag{9b}$$

where

$$\Delta_n^{TM} = x_{n\ell+1}^{TM} - x_{n\ell}^{TM} \tag{9c}$$

and

$$X_{n\ell}^{TM} = x_{n\ell}^{TM} - \tfrac{1}{2}\Gamma_{n\ell}^{TM}. \tag{9d}$$

8. Franz, W., "Über die Greenschen Funktionen des Zylinders und der Kugel", Z.Naturforsch., Vol.A9,p.705,1954;Franz,Walter, and Raimund Galle"Semiasymptotische Reihen fur die Beugung einer ebenen Welle am Zylinder", Z.Naturforsch.,Vol.A10, p.374, 1955.

9. Brill, D., and H. Überall,"Acoustic Waves Transmitted through Solid Elastic Cylinders", J.Acoust.Soc.Amer., Vol.50,p.921, 1971; Gaunaurd, G.C., E.Tanglis, H. Überall and D. Brill, "Interior and Exterior Resonances in Acoustic Scattering I: Spherical Targets", J. Appl.Phys. (to be submitted).

Equations (9a,b) are sufficient to determine ε and δ since $\Delta_{n\ell}$ and $x_{n\ell}$ are provided by the observed resonance locations and widths, so that the inverse problem (i.e. the determination of the properties of the scattering object from the observed properties of the echo) has been solved for the present case. This serves to illustrate the power of the resonance approach as regards a utilization of the information contained in the resonances for purposes of target discrimination, being a power which evidently extends far beyond the simple example that has been analyzed here[10,11] [4.11, 4.16, 4.19].

We wish to thank Messrs. P. J. Moser, J. D. Murphy, and A.Nagl for their contributions. The support of the Naval Air Systems Command, AIR-310B, and of the Independent Research Board of NSWC is acknowledged.

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10. Gaunaurd, G. C., K. P. Scharnhorst, and H. Überall, "New Method to Determine Shear Absorption using the Viscoelastodynamic Resonance-Scattering Formalism", J. Acoust. Soc. Amer., Vol. 64, p. 1211, 1978.

11. Guanaurd, G. C., and H. Überall, "Deciphering the Scattering Code Contained in the Resonance Echoes from Fluid-Filled Cavities in Solids", SCIENCE, Vol. 206, p. 61, 1979.

RADAR ECHO ANALYSIS BY THE SINGULARITY EXPANSION METHOD

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ABSTRACT

Pulse mode radar operation is analyzed under the assumption that the scattering object Γ lies in the far field of both the transmitter and the receiver. It is shown that, in this approximation, the radar signal is a plane wave $s(x \cdot \theta_0 - t, \theta_0)$ near Γ, where θ_0 is a unit vector directed from the transmitter toward Γ, and similarly the echo is a plane wave $e(x \cdot \theta - t, \theta, \theta_0)$ near the receiver, where θ is a unit vector directed from Γ toward the receiver. Moreover, it is shown that

$$e(\tau,\theta,\theta_0) = \mathrm{Re}\left\{\int_0^\infty e^{i\tau\omega}\, T(\omega\theta,\omega\theta_0)\, \hat{s}(\omega,\theta_0)\,d\omega\right\}$$

where $\hat{s}(\omega,\theta_0)$ is the Fourier transform of $s(\tau,\theta_0)$ and $T(\omega\theta,\omega\theta_0)$ is the scattering amplitude in the direction θ due to the scattering by Γ of a CW mode plane wave with frequency ω and propagation direction θ_0. Finally the singularity expansion method is used to show that

$$e(\tau,\theta,\theta_0) \sim \sum e^{i\tau\omega_n}\, T_n(\theta,\theta_0)\, \hat{s}(\omega_n,\theta_0), \quad \mathrm{Im}\,\omega_n < 0 \ .$$

1. INTRODUCTION - RADAR ECHO PREDICTION

This paper presents an application of C. E. Baum's singularity expansion method (SEM) [2.1] and the author's method of *asymptotic wave functions* [6.115, 6.119, 6.120] to the prediction of pulse mode radar echoes from bounded scatterers. The results presented here are generalizations of corresponding results for sonar echoes [6.117]. Only a summary of the principal concepts and results is presented here. A complete exposition of the theory is planned for a separate publication.

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1.1 Physical Assumptions

Radar echo structure is analyzed below under the following assumptions. The radar system (transmitter and receiver) operates in a stationary homogeneous isotropic unlimited medium. The system is stationary with respect to the medium. The scatterers are bounded perfectly conducting objects. The scatterers are stationary with respect to the medium. The transmitter and receiver are in the far field of the scattering objects. In addition it is assumed that secondary echoes due to the radar system components are negligible.

1.2 Mathematical Formulation

A fixed Cartesian coordinate system is used throughout the paper. $x = (x_1, x_2, x_3) \in R^3$ denotes a coordinate triple of this system and $t \in R$ denotes a time coordinate. Γ denotes a closed bounded subset of R^3 that represents the scatterers and $\Omega = R^3 - \Gamma$ denotes the domain exterior to Γ. The common frontier of Γ and Ω, which represents the surface of the scatterers, is denoted by $\partial\Omega$. The medium filling Ω is characterized by a dielectric constant ε and a magnetic permeability μ. It will be assumed that $\varepsilon = 1$ and $\mu = 1$ since this can be achieved by a suitable choice of units.

The electric and magnetic fields will be described by their components, (E_1, E_2, E_3) and (H_1, H_2, H_3) respectively, relative to the fixed Cartesian system. It will be convenient to use the notation and conventions of matrix algebra, rather than vector algebra, and to characterize the electromagnetic field by the 6×1 column matrix

$$u = u(t,x) = (E_1\ E_2\ E_3\ H_1\ H_2\ H_3)^T \tag{1.1}$$

where M^T denotes the transpose of matrix M. Similarly, if the electric and magnetic current densities that generate the field are described by their components, (J_1, J_2, J_3) and (J_1', J_2', J_3') respectively, then

$$f = f(t,x) = (J_1\ J_2\ J_3\ J_1'\ J_2'\ J_3')^T \tag{1.2}$$

characterizes the field sources. With these conventions Maxwell's field equations can be written

$$D_t u + \sum_{j=1}^{3} A_j D_j u + f = 0 \quad \text{for} \quad t \in R,\ x \in \Omega \tag{1.3}$$

where $D_t = \partial/\partial t$, $D_j = \partial/\partial x_j$ $(j = 1,2,3)$ and A_1, A_2, A_3 are the three symmetric 6×6 matrices defined by

$$\sum_{j=1}^{3} A_j p_j = \begin{pmatrix} 0 & M(p) \\ -M(p) & 0 \end{pmatrix}, \quad M(p) = \begin{pmatrix} 0 & p_3 & -p_2 \\ -p_3 & 0 & p_1 \\ p_2 & -p_1 & 0 \end{pmatrix} . \tag{1.4}$$

The field equations (1.3) will be supplemented by the boundary condition for a perfect electrical conductor. It can be written

$$M(n)\, E = 0 \quad \text{on} \quad \partial\Omega \tag{1.5}$$

where $n = (n_1, n_2, n_3)$ is a unit normal vector on $\partial\Omega$ and $E = (E_1\ E_2\ E_3)^T$ is the electric part of u.

A theory of solutions with finite energy of (1.3), (1.5) was given in [6.118]. The total field energy at time t is given by

$$E = \frac{1}{2} \int_\Omega u(t,x)^T\, u(t,x)\, dx \tag{1.6}$$

where $dx = dx_1\, dx_2\, dx_3$. The theory of [6.118] makes use of the energy norm

$$\|u\|_\Omega = \left\{\frac{1}{2} \int_\Omega u(x)^T\, u(x)\, dx\right\}^{1/2} \tag{1.7}$$

and corresponding Hilbert space H. The pulse mode radar echoes constructed below are in H.

2. PULSE MODE RADAR SIGNAL STRUCTURE

The transmitter will be assumed to be localized in the ball $B(x_0, \delta_0) = \{x\colon |x - x_0| < \delta_0\}$ and to act during an interval $0 < t < t_0$. The corresponding pulse mode radar signal is the electromagnetic field $u_0(t,x)$ that is generated by f when no scatterers are present. Thus u_0 is characterized by the conditions

$$D_t u_0 + \sum_{j=1}^{3} A_j D_j u_0 + f = 0 \quad \text{for} \quad t \in R,\ x \in R^3 , \tag{2.1}$$

$$u_0(t,x) = 0 \quad \text{for} \quad t < 0,\ x \in R^3 . \tag{2.2}$$

The field u_0 can be constructed by Fourier analysis or by the method of retarded potentials [6.115, 6.117, 6.119] but these constructions will not be used here.

2.1 Asymptotic Wave Fields

For definiteness the scatterers are assumed to be localized in the ball $B(0,\delta)$ centered on the origin: $\Gamma \subset B(0,\delta)$. As a normalization it is assumed that $\delta \approx 1$ and $\delta_0 \approx 1$. With this convention the assumption that the transmitter lies in the far field of Γ can be formulated as $|x_0| >> 1$. The signal, propagating at the speed $c = (\varepsilon\mu)^{-1/2} = 1$, will arrive at Γ at a time t of the same magnitude as $|x_0|$, whence $t >> 1$.

It was shown in [6.115] that each signal u_0 with finite energy has an asymptotic wave field u_0^∞ of the form

(2.3) $u_0^\infty(t,x) = |x-x_0|^{-1}\, s(|x-x_0| - t,\theta)$, $\theta = (x-x_0)/|x-x_0|$

such that

(2.4) $$\lim_{t\to+\infty} \|u_0(t,\cdot) - u_0^\infty(t,\cdot)\|_{R^3} = 0 \quad .$$

The wave profile $s(\tau,\theta)$ is defined for all $(\tau,\theta) \in R \times S^2$ where S^2 is the unit sphere in R^3. Moreover, by specializing the results of *[6.115] it can* be shown that $s(\tau,\theta)$ has the properties

(2.5) $$\int_R \int_{S^2} s(\tau,\theta)^T\, s(\tau,\theta)\; d\theta\, d\tau < \infty \quad ,$$

where $d\theta$ is the element of area on S^2 (solid angle), and

(2.6) $$P(\theta)\; s(\tau,\theta) = s(\tau,\theta)$$

where

(2.7) $$P(\theta) = \frac{1}{2}\begin{pmatrix} 1 - \theta\theta & M(\theta) \\ -M(\theta) & 1 - \theta\theta \end{pmatrix} \quad \text{for all} \quad \theta \in S^2 \quad .$$

In (2.7), $\theta\theta$ denotes the dyadic, or tensor, product of θ with itself with components $\theta_j\theta_k$. Property (2.6), (2.7) characterizes the polarization properties of the asymptotic wave fields u_0^∞.

The function $s(\tau,\theta)$ will be called the pulse mode transmitter radiation pattern. It can be constructed from the source function f ; see [6.117]. However, it will be assumed here that s, rather than f, is given since s is the important function in pulse mode transmitter design. The construction of a transmitter with a prescribed radiation pattern is the task of the transmitter design engineer.

2.2 The Plane Wave Approximation

Define $\theta_0 \in S^2$ by $x_0 = -|x_0|\,\theta_0$. Then θ_0 is directed from the transmitter toward the scatterers and for x near Γ one has

(2.8) $$|x - x_0| = |x_0| + \theta_0 \cdot x + O(|x_0|^{-1}) \quad \text{for} \quad |x_0| >> 1 \quad .$$

Hence, by (2.3),

(2.9) $$u_0^\infty(t) = |x_0|^{-1}\, s(\theta_0 \cdot x - t + |x_0|,\theta_0) + O(|x_0|^{-2})$$

near Γ . If the error term is dropped one has a pulse mode plane wave signal. This approximation is made in the remainder of the paper.

3. PULSE MODE PLANE WAVE SCATTERING

A plane wave signal

$$u_0(t,x) = s(x \cdot \theta_0 - t, \theta_0) \ , \ \text{supp } s(\cdot,\theta_0) \subset [a,b] \ , \tag{3.1}$$

is assumed where the wave profile $s(\tau,\theta_0)$ satisfies

$$P(\theta_0)\, s(\tau,\theta_0) = s(\tau,\theta_0) \ . \tag{3.2}$$

Such a field is a solution of Maxwell's equations (2.2) with $f = 0$. The total field $u(t,x)$ resulting from the interaction of $u_0(t,x)$ with the scatterers is characterized by the properties

$$D_t u + \sum_{j=1}^{3} A_j D_j u = 0 \quad \text{for} \quad t \in R,\ x \in \Omega \ , \tag{3.3}$$

$$M(n)\, E = 0 \quad \text{for} \quad t \in R,\ x \in \partial\Omega \ , \tag{3.4}$$

$$u(t,x) = u_0(t,x) \quad \text{for} \quad t + b + \delta < 0,\ x \in \Omega \tag{3.5}$$

where $E = (u_1\ u_2\ u_3)^T$ is the electric part of u . The scattered field, or echo, is defined by

$$u_e(t,x) = u(t,x) - u_0(t,x) \quad \text{for} \quad t \in R,\ x \in \Omega \ . \tag{3.6}$$

The author has shown, by the method of [6.117, 6.119], that u_e has an asymptotic wave field

$$u_e^\infty(t,x) = |x|^{-1}\, e(|x| - t,\theta,\theta_0),\ x = |x|\ \theta \tag{3.7}$$

that converges to $u_e(t,x)$ in energy when $t \to \infty$:

$$\lim_{t\to+\infty} \| u_e(t,\cdot) - u_e^\infty(t,\cdot) \|_{\mathcal{H}} = 0 \ . \tag{3.8}$$

The proof follows that for the scalar case of [6.117].

Points x in the far field of Γ satisfy $|x| \gg 1$. The echo u_e will arrive at a receiver at such a point when $t \gg 1$. Hence the echo may be approximated in the far field by the asymptotic field (3.7). For this reason $e(\tau,\theta,\theta_0)$ will be called the echo waveform. It depends on the direction of incidence of the plane wave (3.1) and the direction of observation θ. In this approximation, the echo prediction problem is the problem of constructing $e(\tau,\theta,\theta_0)$ when the transmitter radiation pattern $s(\tau,\theta_0)$ and the scatterers Γ are given. The solution to this problem given below is based on the theory of CW mode radar echoes outlined in the next two sections.

4. CW MODE SIGNAL STRUCTURE

The CW mode electromagnetic fields are solutions of the field equations (1.3) of the form

$$u(t,x) = e^{-i\omega t}\, v(x), \quad f(t,x) = e^{-i\omega t}\, \rho(x) \tag{4.1}$$

whence

$$\sum_{j=1}^{3} A_j D_j v - i\omega v = \rho \quad . \tag{4.2}$$

CW mode signals in R^3 are generated by the Green's matrix [6.88]

$$G(x,x',\omega) = \begin{pmatrix} \nabla\nabla + \omega^2 1_3 & -i\omega M(\nabla) \\ i\omega M(\nabla) & \nabla\nabla + \omega^2 1_3 \end{pmatrix} \frac{e^{i\omega|x-x'|}}{4\pi\omega|x-x'|} 1_6 \quad , \tag{4.3}$$

where 1_n denotes the $n \times n$ unit matrix. G is the outgoing solution of the equation

$$\left(\sum_{j=1}^{3} A_j D_j - i\omega \right) G(x,x',\omega) = \delta(x - x')\, 1 \quad . \tag{4.4}$$

The outgoing solution in R^3 of (4.2) is

$$v(x) = \int_{R^3} G(x,x',\omega)\, \rho(x')\, dx' \quad . \tag{4.5}$$

Asymptotic evaluation of $v(x)$ for large $|x|$ using (4.3) and (4.5) gives the far field form

(4.6) $$v(x) = (2\pi)^{1/2}\, \omega\, \frac{e^{i\omega|x|}}{|x|} P(\theta)\, \hat{\rho}(-\omega\theta) + O(|x|^{-2})$$

where $x = |x|\theta$, $P(\theta)$ is defined by (2.8) and

(4.7) $$\hat{\rho}(p) = \frac{1}{(2\pi)^{3/2}} \int_{R^3} e^{-ip\cdot x} \rho(x)\, dx$$

is the Fourier transform of $\rho(x)$. In particular, noting that $P(-\theta)\, P(\theta) = 0$, it is seen that the Silver Müller radiation condition for $v(x)$ can be written

(4.8) $$P(-\theta)\; v(|x|\theta) = O(|x|^{-2}),\ |x| \to \infty \ .$$

4.1 CW Mode Plane Waves

$G(x,x',\omega)$ represents a CW spherical wave from a point source at the point x' . On putting $x' = -|x'|\eta$ in (4.3) and making $|x'| \to \infty$ with x fixed one finds after a short calculation

(4.9) $$G(x,x',\omega) = (2\pi|x'|)^{-1}\, \omega\, e^{i\omega|x'|}\, e^{i\omega\eta\cdot x} P(\eta) + O(|x'|^{-2}) \ .$$

Dropping the error term gives a matrix CW mode plane wave electromagnetic field. The general CW mode plane wave field is obtained by applying (4.9) to a constant vector and dropping the error term. It has the form

(4.10) $$v(x) = e^{ip\cdot x} P(\eta)c,\ p = |p|\eta$$

where c is an arbitrary 6-component vector. This may also be derived from (3.1), (3.2) by taking $s(\tau,\eta) = e^{i\omega\tau} P(\eta)c$. (4.10) is equivalent to the familiar formulas

(4.11) $$E(x) = e^{ip\cdot x}\, \alpha,\ H(x) = e^{ip\cdot x}(\eta \times \alpha),\ \alpha \cdot \eta = 0$$

where $v = (E_1\ E_2\ E_3\ H_1\ H_2\ H_3)^T$, $p = |p|\eta$ and $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ is an arbitrary vector.

5. CW MODE ECHO STRUCTURE

The columns of the 6×6 matrix-valued function

(5.1) $$\psi^0(x,p) = (2\pi)^{-3/2}\, e^{ip\cdot x} P(\eta),\ p = |p|\eta \ ,$$

are CW mode plane waves of the form (4.10). The scattering of the CW mode matrix plane wave (5.1) by Γ produces a CW mode matrix-valued field

$$(5.2) \qquad \psi(x,p) = \psi^0(x,p) + \psi^{sc}(x,p),\ x \in \Omega,\ p \in R^3 - \{0\}$$

that is characterized by the properties

$$(5.3) \qquad \left(\sum_{j=1}^{3} A_j D_j - i|p| \right) \psi(x,p) = 0,\ x \in \Omega \quad ,$$

$$(5.4) \qquad M(n)\ \psi_E(x,p) = 0 \ ,\ x \in \partial\Omega$$

$$(5.5) \qquad P(-\theta)\ \psi^{sc}(|x|\theta,p) = O(|x|^{-2}),\ |x| \to \infty\ ,$$

where ψ_E is the electric part of ψ (a 3×6 matrix). The author has shown the existence and uniqueness of $\psi(x,p)$ for a large class of domains Ω, including the "cone domains" of N. Weck [6.114] and domains having S. Agmon's "restricted cone property" [6.1]. The proofs, which generalize the results of [6.119] to Maxwell's equations, are based on compactness results of N. Weck [6.114] and C. Weber [6.113], respectively. In the special case that $\partial\Omega$ is a smooth surface $\psi(x,p)$ can be constructed by the integral equation method described below.

5.1 Far Field Form of CW Mode Echoes

$\psi^{sc}(x,p)$ is the CW mode echo produced by the scattering of $\psi^0(x,p)$ by Γ. An integral representation of ψ^{sc} by the Green's matrix (4.3) can be used to derive the far field form

$$(5.6) \qquad \psi^{sc}(x,p) = \frac{e^{i|p||x|}}{4\pi|x|} T(|p|\theta,p) + O(|x|^{-2})\ ,\ x = |x|\theta \quad ,$$

where $T(p,p')$ is a 6×6 matrix-valued scattering amplitude. The polarization of the echo in the far field is characterized by the property

$$(5.7) \qquad P(\eta)\ T(|p|\eta,|p|\eta') = 0 \quad .$$

5.2 Construction of $T(p,p')$

Define

$$(5.8) \qquad J(x,p) = n(x) \times \psi_H(x,p),\ x \in \partial\Omega \quad ,$$

where ψ_H is the magnetic part of ψ . $J(x,p)$ is the matrix electric current density on $\partial\Omega$ induced by the plane wave ψ^0 . The divergence theorem and the jump relations of potential theory can be used to show that

$$(5.9) \qquad J(x,p) = 2(n \times \psi_H^0(x,p)) + \int_{\partial\Omega} K(x,x',|p|)\, J(x',p)\, dS'$$

where K is the 3 × 3 matrix-valued kernel

$$(5.10) \qquad K(x,x',\omega) = \frac{1}{2\pi}\left\{ \nabla \frac{e^{i\omega|x-x'|}}{|x-x'|}\, n(x) \cdot - \frac{\partial}{\partial n} \frac{e^{i\omega|x-x'|}}{|x-x'|}\, 1_3 \right\} .$$

If $\partial\Omega$ is smooth then (5.9) is a Fredholm equation and can be used to construct $J(x,p)$ and $\psi(x,p)$; cf. L. Marin and R. W. Latham [3.7, 3.8] and L. Marin [3.9, 3.10]. The scattering amplitude can be calculated from $J(x,p)$ and the relation

$$(5.11) \quad T(p,p') = (2\pi)^{3/2}\, 2i|p| \int_{\partial\Omega} \psi^0(x,p)^* \begin{pmatrix} J(x,p') \\ 0 \end{pmatrix} dS , \quad |p| = |p'| .$$

6. PULSE MODE RADAR ECHO STRUCTURE

The solution of the pulse mode radar echo prediction problem formulated in §3 is given by the relation

$$(6.1) \qquad e(\tau,\theta,\theta_0) = \mathrm{Re}\left\{ \int_0^\infty e^{i\tau\omega}\, T(\omega\theta,\omega\theta_0)\, \hat{s}(\omega,\theta_0)\, d\omega \right\}$$

where

$$(6.2) \qquad \hat{s}(\omega,\theta_0) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-i\omega\tau}\, s(\tau,\theta_0)\, d\tau$$

is the Fourier transform of $s(\tau,\theta_0)$. Thus under the far field assumptions of §1 the echo waveform is determined by the transmitter waveform and the matrix scattering amplitude $T(\omega\theta,\omega\theta_0)$. The latter can be calculated by solving the integral equation (5.9) an . using relation (5.11).

Equation (6.1) is the generalization to electromagnetic fields of the analogous result for acoustic scattering that was derived in [6.117]. A proof of (6.1) may be given by the method of [6.117]. The key item in the proof is the theorem that the CW mode fields $\psi(x,p)$ are a complete family of generalized eigenfunctions for the Maxwell system. A proof along the lines of [6.119] may be based on the results of Weck [6.114] or Weber [6.113].

7. SEM EXPANSION OF PULSE MODE RADAR ECHOES

If the scatterers Γ are bounded by smooth surfaces the integral equation (5.9) can be solved for $J(x,\omega\theta)$ by the Fredholm determinant method [6.116]. Note that $\psi^0(x,\omega\theta)$ and $K(x,x',\omega)$ are entire functions of ω. It follows from the Fredholm theory that

$$(7.1) \qquad J(x,\omega\theta) = \frac{M(x,\omega\theta)}{D(\omega)}$$

and hence

$$(7.2) \qquad T(\omega\theta,\omega\theta_0) = \frac{N(\omega\theta,\omega\theta_0)}{D(\omega)}$$

where $D(\omega)$, $M(x,\omega\theta)$ and $N(\omega\theta,\omega\theta_0)$ are entire functions of ω. Moreover, the poles of $T(\omega\theta,\omega\theta_0)$ can be shown to lie in the lower half-plane. These facts can be used to develop an SEM expansion of the echo waveform (6.1).

The reality of $s(\tau,\theta_0)$ and symmetry properties of $T(p,p')$ imply that (6.1) can be rewritten

$$(7.3) \qquad e(\tau,\theta,\theta_0) = \frac{1}{2}\int_{-\infty}^{\infty} e^{i\tau\omega}\, T(\omega\theta,\omega\theta_0)\, \hat{s}(\omega,\theta_0)d\omega \quad .$$

It is natural to regard this integral as a contour integral in the ω-plane and to shift the contour to a line $\mathrm{Im}\ \omega = -b < 0$. Assume that the poles ω_n of $T(\omega\theta,\omega\theta_0)$ satisfy

$$(7.4) \qquad D'(\omega_n) \neq 0, n = 1,\ 2,\ 3,\cdots$$

$$(7.5) \qquad \{n : -b \leq \mathrm{Im}\ \omega_n < 0\} \text{ is finite}$$

$$(7.6) \qquad |N(\omega\theta,\omega\theta_0)| \leq C|\omega|^m \text{ for } -b \leq \mathrm{Im}\ \omega \leq 0$$

where C and m are constants. Then (7.3) implies

$$(7.7) \qquad e(\tau,\theta,\theta_0) = \sum_{\mathrm{Im}\ \omega_n \geq b} e^{i\tau\omega_n}\, T_n(\theta,\theta_0)\, \hat{s}(\omega_n,\theta_0) + O(e^{b\tau})$$

where

(7.8) $$T_n(\theta,\theta_0) = -\pi i \operatorname*{Res}_{\omega_n} T(\omega\theta,\omega\theta_0) .$$

Hypothesis (7.4) is inessential. If $T(\omega\theta,\omega\theta_0)$ has a higher order pole then in (7.7) $e^{it\omega_n}$ will be multiplied by a polynomial in τ. Hypotheses (7.5) and (7.6) are closely connected with the geometry of Γ and the associated question of the exponential decay on bounded sets of the scattered fields. For acoustic scattering there is a considerable literature on these questions: see [2.4] and [6.69] and the literature cited there. Far field natural modes for electromagnetic fields have also been defined and used by C. E. Baum [3.3] and F. M. Tesche [4.49].

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This special issue of Electromagnetics is dedicated to the subject of the Singularity Expansion Method (SEM) - in particular the mathematical aspects of SEM. ~~In fact,~~ the issue forms the proceedings of a meeting, "Mathematical Foundations of the Singularity Expansion Method", held at the Carnahan House of the University of Kentucky in November 1980, under the sponsorship of the Air Force Office of Scientific Research (AFOSR). The purpose of the meeting was to bring together a group of mathematicians and engineers who have worked on different aspects of the SEM to foster interdisciplinary communication (CONTINUED)

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ITEM #20, CONTINUED: between the groups. The hope, of course, was that this communication might lead to the resolution of some questions regarding the mathematical rigor that have persisted throughout the development of the SEM. This communication we believe certainly led to a better understanding between the two groups of what the important questions are and the available means of attacking these questions.

Ten years have passed since Carl Baum first formalized the SEM as a means of treating transient and broad band electromagnetic scattering problems [3.1].* This development was sparked by the results from many experiments where different scatterers were exposed to transient electromagnetic fields. It was observed during these experiments that the response of the scatterer appeared to consist of a superposition of damped sinusoidal oscillations whose frequencies are related to the size of the scatterer. The natural question that arose was: "Is it possible to express any external scattering response as a sum of damped oscillations whose resonances and damping constants only depend on a cavity?" The SEM was developed when trying to answer this question.

Much work during the last ten years has gone into trying to put the SEM on a solid mathematical foundation and applying it to various scattering problems. Workers who have tried to solidify the mathematical foundations for the method have found a great deal of frustration in dealing with such issues as space-time problems, nonself-adjoint operators, and analytic function theory. There are few general mathematical results which define the SEM representation within the confines of well defined mathematical and physical constraints. In many cases, workers have had to make whatever observations they can from the solution of a specific problem and then extend these results using their physical/mathematical intuition. The wealth of semiempirical data acquired this way nevertheless have resulted in heuristically derived rules for the applicability and validity of the SEM. Thus, even in the face of the persistent difficulties in developing general theory, SEM stands as a powerful tool in electromagnetic and acoustic scattering theory. The strength of the SEM primarily rests with the fact that both transient and time harmonic scattering quantities can be represented as a sum of conveniently factored products. One factor in this product depends only on the scatterer itself whereas the other depends on the exciting (or incident field.) The quantities that enter into the object-dependent factor are the object's complex resonant frequencies and the associated natural mode currents. The constellation of natural frequencies can be used to characterize the scattering object, thus opening the possibility of using SEM for target classification purposes. The expansion of the object's response in terms of natural modes allows for a circuit description of certain EM properties of the object. The discussions during the meeting in the Carnahan House reflected the differences in the mathematician's and engineer's outlooks. A mathematician participant was careful to categorize his comments into 'results' (conclusions which can be mathematically proven) and 'observations' (conclusions drawn from special cases but not proven mathematically). Engineers were quick to state that a significant part of their SEM related activity is predicated upon 'observations' only (as is so much of their overall work). As a consequence the papers contained in this issue can perhaps be described as a collection of 'results' and 'observations'. We leave it to the reader to distinguish between 'results' and 'observations' and the relative merit of the two.

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